This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

45. 1504.

45. 1504.

PLANE TRIGONOMETRY

AND

MENSURATION.

BIBLIOTHECA BODLEIANA

For the Use of the Royal Military College.

BY

WILLIAM SCOTT, M.A. F.R.A.S.

PROFESSOR OF MATHEMATICS IN THE INSTITUTION.

LONDON:
PRINTED FOR
LONGMAN, BROWN, GREEN, AND LONGMANS,
PATERNOSTER-ROW.
1845.

London:
Printed by A. Spottiswoode,
New-Street-Square.

ADVERTISEMENT.

THE following Treatise of Plane Trigonometry and Mensuration is the third of a series which is to constitute a General Course of Mathematics for the use of the gentlemen cadets and the officers in the senior department of this Institution. The Course, when completed, will comprehend the subjects whose titles are subjoined: — I. Arithmetic and Algebra.* II. Geometry.* III. Plane Trigonometry with Mensuration.* IV. Analytical Geometry with the Differential and Integral Calculus, and the Properties of Conic Sections. V. Practical Astronomy and Geodesy, including Spherical Trigonometry.* VI. The Principles of Mechanics, and VII. Physical Astronomy.

Royal Military College,
1845.

---

* Published.

PREFACE.

THE direct object of Plane Trigonometry is the discovery of the relations which subsist among the sides and angles of rectilineal triangles. Whether the investigation of these relations is conducted geometrically or analytically, the values of the results immediately deduced from them are numerical. The processes also of deducing the expressions of the results, and of combining them to obtain formulæ for the solutions in complex propositions, are algebraical or arithmetical.

The applications of Trigonometry in the higher branches of mathematical and physical science are purely analytical. For these reasons, and to preserve uniformity of method, it has become the usual practice to treat the whole subject of Plane Trigonometry analytically.

In Mensuration, the general problem is, to determine the relation which any proposed geometrical magnitude bears to an arbitrary unit of the same kind. This relation is numerical. If general, it is expressed by algebraic symbols: if particular, by a number. The processes of combining the quantities and deducing the values of the results, as in plane trigonometry, are algebraical or arithmetical. Hence the analytical seems likewise the most convenient method of investigating those relations of magnitude which are contemplated in Mensuration. It is, therefore, adopted in the sections relating to this subject, as well as in those which relate to trigonometry, in the following treatise.

The whole work is divided into nine Sections.

Of these, the first and second are devoted to the consideration of the manner of expressing geometrical magnitudes by numbers. The investigations of a considerable number of important trigonometrical formulæ and series are given in the third. The fourth contains rules for the computation of the trigonometrical tables.

The application of certain formulæ of Section III. to the resolution of plane rectilineal triangles, and the calculation, by logarithms, of particular examples in each of the cases of plane trigonometry, are contained in Section V.

In Section VI. are described the instruments generally used in the measurement of heights and distances, and in surveying.

Section VII. contains applications of trigonometry to the measurement of heights and distances, and the determination of the positions of stations by angles taken at the extremities of a measured base; and also by angles taken, at the stations themselves, between objects whose positions are previously known.

Formulæ for the calculation of the areas of plane figures bounded by straight lines and the circumferences of circles are given in Section VIII.

Section IX. contains similar formulæ for the calculation of the areas of the surfaces and the volumes of solid figures bounded by plane surfaces, and also of the cylinder, cone, and sphere.

Subjoined is an Appendix containing notes and related matter which might have seemed out of place in the text.

In the formulæ of Sections V., VII., VIII., and IX., the values of the required or unknown quantities are expressed in terms of the original data of the respective problems. Generally, the formulæ are adapted to logarithmic calculation; and the manner of applying them to particular cases is, for the most part, illustrated by one or more examples.

Certain paragraphs and articles which seem the least elementary or important are printed in a smaller type than the rest of the work. Throughout the volume also, many equations and formulæ of considerable length are printed in small type, that they may be contained in one line. These, however, are easily distinguishable from the former.

ERRATA.

Page.	Line.	Error.	Correction.
12	2. from foot	eq.	proportion
37	14. from foot	coincides A X	coincides with A X
40	last line	tan. A	cot. A
41	running title	COS. A	COT. A
44	12. from top	arc. B D	arc B D
52	10. from top	cos. 30 sin. 30	cos. 30° sin. 30°
60	7. from top	factor	factors
—	3. from foot	$(\cos.\phi + \sqrt{-1}\sin.\phi)^{-}$	$(\cos.\phi + \sqrt{-1}\sin.\phi)^{-n}$
67	7. from foot	${}^{\varepsilon}{}_{x}\sqrt{-1}$	$\varepsilon^{x\sqrt{-1}}$
—	5. from foot	$\varepsilon^{2\sqrt{-1}}$	$\varepsilon^{2x\sqrt{-1}}$
68	7. from top	$\frac{1}{n-^{4}}$ + &c.	$\frac{1}{\theta^{n-4}}$ + &c
69	4. from top	to cos. $0x$,	to cos. $0x$, together with the term cos. $0x$,
—	6. from top	the term containing cos. $0x$ is the middle term:	the co-efficient of x in the $\frac{n+1}{2}$ and the $\frac{n+1}{2}+1$ terms is 1:
—	7. from top	the preceding terms together with the middle term,	the terms from cos. nx to cos. x inclusive,
70	5. from top	quality	quantity
85	last line	$\frac{1}{3}$	$-\frac{1}{3}$M
91	14. from top	$-a$	$-a^2$
93	12. from top	sin. $\frac{a}{2}$	sin. $\frac{A}{2}$
100	2. from top	log. sin. a	log. sin. A
101	16. from foot	tan. $\frac{1}{2}$(B − c)	tan. $\frac{1}{2}$(B − C)
102	3. from top	+ arith. comp. of 882	+ arith. comp. of log. 882
128	8. from foot	base A D	base A B
131	8. from foot	$\frac{c \sin. D'}{a \sin. D \sin. (D + D')}$	$\frac{c \sin. D}{a \sin. D \sin. (D + D')}$
144	7. from top	18 . a	8 . a
151	15. from top	$\frac{2 \sin. A \cos. A}{\sin. A \sin. D}$	$\frac{2 \sin. A \cos. A}{\sin. A \sin. A}$
169	10. from foot	$r = \frac{c^2 + 4v}{8v}$	$r = \frac{c^2 + 4v^2}{8v}$
170	5. from foot	2.2274	2.2474
174	4. from foot	$= 18$	$= \frac{18}{2}$
194	Fig. 1.	A B′C′	A′B′C′
195	14. and 29. from top	plane	plan
196	upper Fig.	F P G H	F P Q H
210	11. from foot	plane	plan
212	11. and 12. from foot	A B	B C
214	18. from top	bearing co	bearing of co
216	13. from top	carries	carry
—	Fig.	A′C′F	A′C′F′
218	7. from top	$= -\sin. a$	$= -\frac{1}{4}\sin. a$
221	16. from top	$\left(\frac{1}{3} + \frac{2}{3z}\right)$	$\left(\frac{1}{3} - \frac{2}{3z}\right)$
222	Fig.	A C	A C E
223	14. from foot	c, c'	C, C′
226	Fig.	F $n'n$	F $n'h$
230	7. from foot	is	are

CONTENTS.

SECTION I.

OF THE NUMERICAL VALUES OF STRAIGHT LINES, PARALLELOGRAMS, TRIANGLES, PRISMS, AND PYRAMIDS. — FIRST PRINCIPLES OF THE APPLICATION OF ALGEBRA TO GEOMETRY. — RATIO OF THE DIAMETER TO THE CIRCUMFERENCE OF A CIRCLE.

Art. Page
1. To find a straight line which shall be equal to two given straight lines, A, B; - - - - - - 1
   a. or to any number of given straight lines; - - 1
   b. or to any multiple of a given straight line - - 1
2. To find a straight line which shall be equal to the excess of a given straight line A, over another given straight line B - 1
   a. Case of $B = A$ - - - - - - - 2
   b. Case of $B > A$: negative line - - - - 2
   c. Direction of a positive line contrary to that of a negative line - - - - - - - 2
   d. The line $A - B$, in its change of sign, passes through zero 2
   e. To find a straight line which shall be equal to the excess of a given straight line over two or more given straight lines - - - - - - - 2
   f. To find a straight line which shall be equal to the excess of a given straight line over any multiple of another given straight line - - - - - - - 2
   g. To find the difference between any given straight lines A, B, C, &c., and any other given straight lines H, K, L, &c. 2
3. Rule for the change of sign of a line - - - 2
4. To find the expression of the numerical value of a given straight line, in terms of an arbitrary unit of length - - 3
   a. The process involves the investigation of the greatest common measure of two lines - - - 4
   b. Case in which the proposed lines are commensurable - 4
   c. Case in which the proposed lines are incommensurable - 4
5. General investigation of the ratio of two straight lines - 4
6. Methods of measuring the length of any proposed straight line 5
7. a......h. Of the expression of the relations of geometrical magnitudes by numbers - - - - - 6
8. Of the representation of geometrical magnitudes by algebraic symbols, and the translation of the conditions of a geometrical proposition into the language of algebra - 8
   a. Area of a triangle in terms of the three sides - - 9

Art. Page
Ratio of the diameter to the circumference of a circle - 11
b. Every circumference contains its diameter the same number of times. This number ($=3.1415926,$) is denoted by π 15
c. If circumference $=1$; diameter $=\cdot3183099$ - - 15
d. Of the geometrical construction of algebraic expressions - 16
9. List of the geometrical propositions from which equations are commonly derived - - - - - - 16
a. The terms of equations, derived from geometrical propositions, are homogeneous - - - - 16
b. Case of exception - - - - - - 17

SECTION II.

OF THE NUMERICAL VALUES OF ANGLES AND CIRCULAR ARCS.

10. To find an angle which shall be equal to the sum of two given angles; - - - - - - - 18
a. or to any multiple of a given angle - - - 18
11. To find an angle which shall be equal to the difference of two given angles; - - - - - - 18
a. or to the difference between one angle and several other angles, or any multiple of another angle - - 18
b. Case in which the angle taken away is the greater. Negative angle - - - - - - - 18
12. Greatest common measure of two angles. Numerical value of an angle - - - - - - - 18
a. Division of the right angle into degrees, minutes, &c. - 19
b. Centesimal division of the right angle. Reduction of sexagesimal degrees, &c., to centesimal degrees, &c. - 19
13. Angular magnitude, unlimited - - - - - 20
a. Negative angles - - - - - - 20
14. Division of the quadrant into degrees, &c. - - 20
a. Correspondence between the measures of angles and circular arcs - - - - - - - 20
b. Relations of the lengths of arcs of the same number of degrees, &c., in circles which have radii of different lengths 20
c. Circular measure $\left(\frac{\text{arc}}{\text{rad.}}\right)$ of an angle - - - 21
d. Value of an arc of a circle expressed in terms of the unit of length by which the radius is measured - - 21
e. Mutual relations of the radius, the angle at the centre of a circle, and the length of the corresponding arc - - 22
f. In every circle, an arc of $57^\circ\ 17'\ 44''.8$ is equal in length to the radius: and an arc of $1''$ is equal to $\frac{10}{2062648}$ths of the radius - - - - - - - 22
g. Manner of expressing the numerical value of an arc or angle, the arc whose length is equal to that of the radius being considered as unity - - - - 22

SECTION III.

TRIGONOMETRICAL FORMULÆ.

Art. Page

15. Definitions of the expressions, complement of an arc or angle, and supplement of an arc or angle - - - 24
16. Notation employed to express the numerical values of the angles, the sides and the ratios of the sides of a triangle. Number of the ratios of the sides - - - 24
    a, b. Use of these ratios in finding the values of the sides of similar triangles - - - - - - 25
    c. In right-angled triangles, if the magnitudes of the acute angles are constant, the ratios of the sides are constant - 25
    d. In right-angled triangles, if the magnitudes of the acute angles are changed, the ratios of the sides are made to *vary* 26
    e. If the ratio of any two sides of a right-angled triangle is made to vary, the magnitudes of the acute angles must vary - - - - - - - 28
    f. Use of the numerical values of the ratios of the sides of right-angled triangles - - - - - - 28
17. Names of the six ratios of the sides of a right-angled triangle 28
18. Changes in the sign and value of the sine of an angle, the angle being made to increase from 0 to 360° - - 29
19. $\sin. A = \sin.(2n.180^\circ + A) = \sin.\{(2n+1)180^\circ - A\} = -\sin.\{(2n+1)180^\circ + A\} = -\sin.(2n.180^\circ - A)$ 33
    a. $\sin. A = \cos.(90^\circ - A)$ - - - - - 33
    b. $\sin. A = \dfrac{1}{\text{cosec.}\, A}$ - - - - - - 33
    c. $\sin. A = \cos. A \tan. A$ - - - 33
    d. $\sin. A = \sqrt{1 - \cos.^2 A}$ - - - - 34
20. Changes in the sign and value of the tangent of an angle, the angle being made to increase from 0 to 360° - - 34
21. $\tan. A = \tan.(2n.180^\circ + A) = -\tan.\{(2n+1)180^\circ - A\} = \tan.\{(2n+1)180^\circ + A\} = -\tan.(2n.180^\circ - A)$ 36
    a. $\tan. A = \cot.(90^\circ - A)$ - - - - 36
    b. $\tan. A = \dfrac{\sin. A}{\cos. A} = \dfrac{\sin. A}{\sqrt{1 - \sin.^2 A}} = \dfrac{\sqrt{1 - \cos.^2 A}}{\cos. A}$ - 36
    c. $\tan. A = \sqrt{\sec.^2 A - 1}$ - - - - 37
    d. $\tan. A = \dfrac{1}{\cot. A}$ - - - - - - 37
22. Changes in the sign and value of the secant of an angle, the angle being made to increase from 0 to 360° - - 37
23. $\sec. A = \sec.(2n.180^\circ + A) = -\sec.\{(2n+1)180^\circ - A\} = -\sec.\{(2n+1)180^\circ + A\} = \sec.(2n.180^\circ - A)$ 39
    a. $\sec. A = \text{cosec.}(90^\circ - A)$ - - - - 39
    b. $\sec. A = \dfrac{1}{\cos. A}$ - - - - - - 39

Art. Page

c. $\sec. A = \frac{1}{\sqrt{1 - \sin.^2 A}}$ - - - - - 39

d. $\sec. A = \sqrt{1 + \tan.^2 A}$ - - - - - 39

24. Limits of the variation of cos. A - - - - - 39

a, b. $\cos. A = \cos. (2n . 180^\circ + A) = -\cos. \{(2n+1)180^\circ - A\} = -\cos. \{(2n+1)180^\circ + A\} = \cos. (2n . 180^\circ - A)$ 40

c. $\cos. A = \sin. (90^\circ - A)$ - - - 40

d. $\cos. A = \frac{1}{\sec. A}$ - - - - - - 40

e. $\cos. A = \frac{1}{\sqrt{1 + \tan.^2 A}}$ - - - - - 40

f. $\cos. A = \frac{\sin. A}{\tan. A} = \frac{\sin. A}{\sqrt{\sec.^2 A - 1}}$ - - - 40

g. $\cos. A = \sqrt{1 - \sin.^2 A}$ - - - - - 40

25. Limits of the variation of cot. A - - - - - 40

a. $\cot. A = \cot. (2n . 180^\circ + A) = -\cot. \{2n+1)180^\circ - A\} = \cot. \{(2n+1)180^\circ + A\} = -\cot. (2n . 180^\circ - A)$ 41

b. $\cot. A = \frac{\cos. A}{\sin. A}$ - - - - - - 41

c. $\cot. A = \tan. (90^\circ - A)$ - - - - - 41

d. $\cot. A = \sqrt{\mathrm{cosec.}^2 A - 1}$ - - - - - 41

26. Limits of the variation of cosec. A - - - - 41

a. $\mathrm{cosec.} A = \mathrm{cosec.} (2n . 180^\circ + A) = \mathrm{cosec.} \{(2n+1)180^\circ - A\} = -\mathrm{cosec.} \{(2n+1)180^\circ + A\}$
$= -\mathrm{cosec.} (2n . 180^\circ - A) = -\mathrm{cosec.} (-A)$ - - - - 42

b. $\mathrm{cosec.} A = \sec. (90^\circ - A)$ - - - - - 42

c. $\mathrm{cosec.} A = \frac{1}{\sin. A} = \frac{1}{\sqrt{1 - \cos.^2 A}}$ - - - - 42

d. $\mathrm{cosec.} A = \sqrt{1 + \cot.^2 A}$ - - - - - 42

27. Sin. A cosec. A = tan. A cot. A = sec. A cos. A = 1 - - 42

28. Expressions for tan. A, sec. A, cos. A, cot. A, cosec. A in terms of sin. A - - - - - - - - 42

29. Limits of the variation of versin. A; coversin. A; suversin. A 42

30. *a......l.* Definitions of the trigonometrical ratios, as straight lines connected with any arc of a circle whose radius is 1 43

l. Transformation of trigonometrical formulæ in which radius $= 1$, into equivalent expressions in which radius $= r$ 45

m.
$\sin.\phi = \cos.\left(\frac{\pi}{2} - \phi\right) = \sin. (2n\pi + \phi) = \sin. \{(2n+1)\pi - \phi\} = -\sin.\{(2n+1)\pi + \phi\} = -\sin.(2n\pi - \phi).$

$\tan.\phi = \cot.\left(\frac{\pi}{2} - \phi\right) = \tan. (2n\pi + \phi) = -\tan. \{(2n+1)\pi - \phi\} = \tan.\{(2n+1)\pi + \phi\} = -\tan.(2n\pi - \phi).$

$\sec.\phi = \mathrm{cosec.}\left(\frac{\pi}{2} - \phi\right) = \sec. (2n\pi + \phi) = -\sec. \{(2n+1)\pi - \phi\} = -\sec.\{(2n+1)\pi + \phi\} = \sec.(2n\pi - \phi).$

$\cos.\phi = \sin.\left(\frac{\pi}{2} - \phi\right) = \cos. (2n\pi + \phi) = -\cos. \{(2n+1)\pi - \phi\} = -\cos.\{(2n+1)\pi + \phi\} = \cos.(2n\pi - \phi).$

$\cot.\phi = \tan.\left(\frac{\pi}{2} - \phi\right) = \cot. (2n\pi + \phi) = -\cot. \{(2n+1)\pi - \phi\} = \cot.\{(2n+1)\pi + \phi\} = -\cot.(2n\pi - \phi).$

$\mathrm{cosec.}\phi = \sec.\left(\frac{\pi}{2} - \phi\right) = \mathrm{cosec.}(2n\pi + \phi) = \mathrm{cosec.} \{(2n+1)\pi - \phi\} = -\mathrm{cosec.}\{(2n+1)\pi + \phi\} = -\mathrm{cosec.} (2n\pi - \phi).$ 45

Art. Page

n. $\tan. 1'' = \text{arc of } 1'' = \sin. 1''$

$\frac{\tan. h}{h} = 1$; $\frac{\sin. h}{h} = 1$; (h denoting an indefinitely small arc) 46

o. $n \tan. \frac{\varphi}{n} = \varphi$; ($n$ being an indefinitely great number) 47

p. The length of the arc of 1″, expressed in parts of the radius $= 1$ is $.0000048481368+$ - - - 47

q. Manner of expressing small arcs or angles in seconds of a degree - - - - - - - 47

31. $\text{Sin.}(A \pm B) = \sin. A \cos. B \pm \cos. A \sin. B$ - - - 47

$\cos.(A \pm B) = \cos. A \cos. B \mp \sin. A \sin. B$ - - - 48

32. $\sin.(A \pm B \pm C) = \sin. A \cos. B \cos. C \pm \sin. B \cos. A \cos. C \pm \sin. C \cos. A \cos. B - \sin. A \sin. B \sin. C$

$\cos.(A \pm B \pm C) = \cos. A \cos. B \cos. C \mp \sin. A \sin. B \cos. C \mp \sin. A \cos. B \sin. C - \cos. A \sin. B \sin. C$ 51

a. If $A + B + C = 180^\circ$; $\sin. A \sin. B \sin. C = \sin. A \cos. B \cos. C + \sin. B \cos. A \cos. C + \sin. C \cos. A \cos. B$ 51

33. *a, b.* $\sin. 60^\circ = \cos. 30^\circ = \frac{\sqrt{3}}{2}$: $\cos. 60^\circ = \sin. 30^\circ = \frac{1}{2}$ - 51

c. $\tan. 60^\circ = \sqrt{3}$ - - - - - 51

d. $\sin. 45^\circ = \cos. 45^\circ = \frac{1}{\sqrt{2}}$: $\tan. 45^\circ = 1$ - - 52

e, f. $\sin. 75^\circ = \cos. 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$: $\cos. 75^\circ = \sin. 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$ 52

34. $\text{Sin.} 2A = 2 \sin. A \cos. A$: $\sin. A = 2 \sin. \frac{A}{2} \cos. \frac{A}{2}$ - - 52

a. $\cos. 2A = \cos.^2 A - \sin.^2 A$: $\cos. A = \cos.^2 \frac{A}{2} - \sin.^2 \frac{A}{2}$ · 53

b. $\sin. 3A = 3 \sin. A \cos.^2 A - \sin.^3 A$ - - - 53

c. $\cos. 3A = \cos.^3 A - 3 \cos. A \sin.^2 A$ - - - 53

d. $\sin. A = 3 \sin. \frac{A}{3} \cos.^2 \frac{A}{3} - \sin.^3 \frac{A}{3}$: $\cos. A = \cos.^3 \frac{A}{3} - 3 \cos. \frac{A}{3} \sin.^2 \frac{A}{3}$ 53

35. $\cos. \frac{1}{2} A = \pm \sqrt{\frac{1 + \cos. A}{2}}$: $\sin. \frac{1}{2} A = \pm \sqrt{\frac{1 - \cos. A}{2}}$ - 53

36. $\cos. A + \sin. A = \pm \sqrt{1 + \sin. 2A}$: $\cos. A - \sin. A = \pm \sqrt{1 - \sin. 2A}$ 54

37. $\tan.(A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A \tan. B}$ - - - - 54

a. $\tan.(A + B + C) = \frac{\tan. A + \tan. B + \tan. C - \tan. A \tan. B \tan. C}{1 - \tan. A \tan. B - \tan. A \tan. C - \tan. B \tan. C}$ 55

b. If $A + B + C = 180^\circ$; $\tan. A + \tan. B + \tan. C = \tan. A \tan. B \tan. C$ 55

38. $\tan. 2A = \frac{2 \tan. A}{1 - \tan.^2 A}$ - - - - - - 55

a. $\tan. 3A = \frac{3 \tan. A - \tan.^3 A}{1 - 3 \tan.^2 A}$ - - - - - 56

Art. Page

b. $\tan.\frac{1}{2}\text{A} = -\frac{1}{\tan.\text{A}} \pm \frac{1}{\tan.\text{A}}\sqrt{1+\tan.^2\text{A}}$ - - 56

c. $\tan.\frac{1}{2}\text{A} = \frac{\sin.\text{A}}{1+\cos.\text{A}} = \frac{1-\cos.\text{A}}{\sin.\text{A}}$ - - - 56

39. $\frac{\sin.(\text{A}+\text{B})+\sin.(\text{A}-\text{B})}{\sin.(\text{A}+\text{B})-\sin.(\text{A}-\text{B})} = \frac{\tan.\text{A}}{\tan.\text{B}} : \frac{\cos.(\text{A}-\text{B})+\cos.(\text{A}+\text{B})}{\cos.(\text{A}-\text{B})-\cos.(\text{A}+\text{B})} = \cot.\text{A}\cot.\text{B}$ 57

40. $\frac{\sin.\text{A}+\sin.\text{B}}{\sin.\text{A}-\sin.\text{B}} = \frac{\tan.\frac{1}{2}(\text{A}+\text{B})}{\tan.\frac{1}{2}(\text{A}-\text{B})}$ - - - - 57

a. $\frac{\cos.\text{A}+\cos.\text{B}}{\cos.\text{B}-\cos.\text{A}} = \cot.\frac{1}{2}(\text{A}+\text{B})\cot.\frac{1}{2}(\text{A}-\text{B})$ - - 57

b. $\frac{\sin.\text{A}\pm\sin.\text{B}}{\cos.\text{A}+\cos.\text{B}} = \tan.\frac{1}{2}(\text{A}\pm\text{B}) : \frac{\sin.\text{A}\pm\sin.\text{B}}{\cos.\text{B}-\cos.\text{A}} = \cot.\frac{1}{2}(\text{A}\mp\text{B})$ 58

41. $\sin.n\text{A}+\sin.(n-2)\text{A} = 2\sin.(n-1)\text{A}\cos.\text{A} : \sin.n\text{A}-\sin.(n-2)\text{A} = 2\cos.(n-1)\text{A}\sin.\text{A}$
$\cos.n\text{A}+\cos.(n-2)\text{A} = 2\cos.(n-1)\text{A}\cos.\text{A} : \cos.(n-2)\text{A}-\cos.n\text{A} = 2\sin.(n-1)\text{A}\sin.\text{A}$ 58

a. Application of the formulæ of this article - - 58

42. $\tan.n\text{A} = \frac{\tan.(n-1)\text{A}+\tan.\text{A}}{1-\tan.(n-1)\text{A}\tan.\text{A}}$ - - - 59

43. De Moivre's formula - - - - - 59

44. $\sin.n\phi = \frac{n}{1}\cos.^{n-1}\phi\sin.\phi - \frac{n(n-1)(n-2)}{1.2.3}\cos.^{n-3}\phi\sin.^3\phi + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos.^{n-5}\phi\sin.^5\phi -$ &c.
$\cos.n\phi = \cos.^n\phi - \frac{n(n-1)}{1.2}\cos.^{n-2}\phi\sin.^2\phi + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos.^{n-4}\phi\sin.^4\phi -$, &c. 62

45. $\sin.x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} +$ &c.
$\cos.x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} +$ &c. - 63

a. $\tan.x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3.3.5.7} +$ &c. - - 64

46. $\phi = \sin.\phi + \frac{1}{2.3}\sin.^3\phi + \frac{1.3}{2.4.5}\sin.^5\phi + \frac{1.3.5}{2.4.6.7}\sin.^7\phi +$ &c. 64

a. $\phi = \frac{\pi}{2} - \cos.\phi - \frac{1}{2.3}\cos.^3\phi - \frac{1.3}{2.4.5}\cos.^5\phi - \frac{1.3.5}{2.4.6.7}\cos.^7\phi -$ &c. 65

b. Meaning of the symbols $\sin.^{-1}x$, $\cos.^{-1}x$, $\tan.^{-1}x$, &c. - 66

47. $\cos.x = \frac{1}{2}(\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}) : \sin.x = \frac{1}{2\sqrt{-1}}(\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}) : \tan.x = \frac{1}{\sqrt{-1}}\left\{\frac{\varepsilon^{2x\sqrt{-1}}-1}{\varepsilon^{2x\sqrt{-1}}+1}\right\}$ 67

a. $\cos.nx = \frac{1}{2}(\varepsilon^{nx\sqrt{-1}} + \varepsilon^{-nx\sqrt{-1}}) : \sin.nx = \frac{1}{2\sqrt{-1}}(\varepsilon^{nx\sqrt{-1}} - \varepsilon^{-nx\sqrt{-1}}) : \tan.nx = \frac{1}{\sqrt{-1}}\left\{\frac{\varepsilon^{2nx\sqrt{-1}}-1}{\varepsilon^{2nx\sqrt{-1}}+1}\right\}$ 67

b. $\sqrt{-1}\tan.x = \frac{\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}}{\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}}$ - - - 68

c. $2^n\cos.^n x = \cos.nx + n\cos.(n-2)x + \frac{n(n-1)}{1.2}\cos.(n-4)x +$ &c. 68

d. $2^n\sin.^n x = \cos.nx - n\cos.(n-2)x + \frac{n(n-1)}{1.2}\cos.(n-4)x -$ &c. 68

$2^n(\sqrt{-1})^{n-1}\sin.^n x = \sin.nx - n\sin.(n-2)x + \frac{n(n-1)}{1.2}\sin.(n-4)x$ - - 68

Art. Page

e. $x = \tan. x - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \frac{\tan.^7 x}{7} + \&c.$ - - 69

48. Values of sin. 18°; cos. 72°; cos. 18°; cos. 36°; sin. 54° - 71

a. Method of verifying trigonometrical formulæ - - 71

SECTION IV.

OF THE CALCULATION OF THE TRIGONOMETRICAL TABLES.

49. Calculation of the values of sin. 1″, cos. 1″, sin. 2″, cos. 2″, &c. - - - - - - - - 73

a, b. Simplifications of the preceding method - - 75

c. The values of the sines of angles greater than 60° are found from the values of the sines of angles less than 60°, by addition - - - - - - - 76

d. The values of the sines and cosines from 30° to 60° are found from the values of the sines and cosines of angles between 0 and 30°, by addition and subtraction - - 77

50. Formulæ for the calculation of the values of the tangents and cotangents of angles from 0 to 90° - - - 78

a. Formulæ for the calculation of the values of the secants and cosecants of angles from 0 to 90° - - - 78

51. Formulæ of verification - - - - - 79

52. Calculation of the values of sin. *x* and cos. *x*, from the formulæ of Article 45. - - - - - 80

a. Definition of the expression, Natural sines - - 81

53. Of the direct calculation of the logarithmic sines and cosines of angles from 0 to 90° - - - - - 81

54. Calculation of the logarithmic sine of an angle of 9° - 87

55. Formulæ for the calculation of the log. tan. and log. cot. of angles - - - - - - - 87

a. Manner of finding the natural sines, &c. from the log. sines, &c. - - - - - - - 87

b. List of the logarithmic tables in common use - - 88

SECTION V.

RESOLUTION OF PLANE RECTILINEAL TRIANGLES.

56. The formulæ for the resolution of right-angled rectilineal triangles are,

$a = c \sin. \text{A}; \; b = c \cos. \text{A}; \; a = b \tan. \text{A};$

a. $c = \frac{a}{\sin. \text{A}} = \frac{b}{\cos. \text{A}}$ - - - - - - 89

57. The relations which subsist between the sides and angles of any rectilineal triangle are expressed by the formulæ $\frac{a}{c} = \frac{\sin. \text{A}}{\sin. \text{C}}$; 89

Art. Page
a. $\frac{a+b}{a-b}=\frac{\tan.\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\tan.\frac{1}{2}(\mathrm{A}-\mathrm{B})}$; - - - - 90
b. $a^2=b^2+c^2-2bc\cos.\mathrm{A}$; - - - - 91
c. $c=\sqrt{a^2+b^2-2ab\cos.\mathrm{C}}$; $\cos.\mathrm{A}=\frac{b^3+c^2-a^2}{2bc}$; - 91
d. $\sin.\frac{\mathrm{A}}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$; - - - 92
e. $\cos.\frac{\mathrm{A}}{2}=\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$; - - - - 92
f. $\tan.\frac{\mathrm{A}}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$; - - - 93
f′. $\tan.\frac{\mathrm{A}}{2}=\frac{1}{s-a}\sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}}=\frac{\mathrm{M}}{s-a}$; - 93
g. $\sin.\mathrm{A}=\frac{2}{bc}\sqrt{\{s(s-a)(s-b)s-c)\}}$ - - - 94
57′. General problem in the resolution of triangles - - 94
a. Difference between the tabular and the real logarithm of a sine, cosine, &c. - - - - - - 95
58. Resolution of the four cases of right-angled triangles - 95
59. Resolution of the four cases of oblique-angled triangles - 98
Remarks on the different solutions of Case IV. Art. 59. - 105

SECTION VI.

MEASUREMENT OF A BASE LINE.—DESCRIPTION OF INSTRUMENTS EMPLOYED IN THE MENSURATION OF HEIGHTS AND DISTANCES, AND IN SURVEYING AND LEVELLING.

60. Measurement of a base line - - - - 107
61. Description and use of the plane table - - - 108
62. Description and use of the theodolite - - - 111
a. Adjustments of the theodolite - - - 114
63. Description and use of the spirit level - - - 116
64. Description and use of the pocket sextant - - - 118
a. Description and use of Gunter's quadrant - - 119
65. Description of the reflecting semicircle - - - 120
66. Description and use of the vernier - - - 121

SECTION VII.

MENSURATION OF HEIGHTS AND DISTANCES.

67. To find the height of an accessible object on a horizontal plane - - - - - - - 123

Art. Page

a. Correction necessary if the angle of elevation is measured with a theodolite - - - - - - 123

68. To find the height of an accessible object on sloping ground 123

69. To find the height and distance of an inaccessible object on a horizontal plane - - - - - - 124

70. To find the height and horizontal distance of an inaccessible object on sloping ground - - - - - 126

71. To find the distance of an inaccessible object from each of two stations, and also from the straight line joining the two stations - - - - - - - 128

72. To find the distance between two inaccessible objects which can be seen from two given stations, in the same plane - 128

a. Converse of the preceding problem - - - 129

73. To determine the position of an object, when there have been observed the angles contained between lines imagined to be drawn from it to three stations whose distances from each other have been previously determined - - - 130

74. To determine the positions of two objects with respect to three stations whose mutual distances are known, when there have been observed, at the place of each object, the angles contained between a line supposed to join the objects, and other lines imagined to be drawn from them to two of the three given stations, some one of these being invisible from each object - - - - 135

SECTION VIII.

MENSURATION OF PLANE SURFACES BOUNDED BY STRAIGHT LINES AND BY THE CIRCUMFERENCES OF CIRCLES.

75. To find the area of a parallellogram,

the base and altitude being given; - - - 139

a. Two sides and the included angle being given - - 140

76. To find the area of a triangle,

the base and altitude being given; - - - 140

a. Two sides and the included angle being given; - - 141

b. Two sides and the angle opposite one of these sides being given; - - - - - - - 142

c. Two angles and one side being given; - - - 142

d. The three sides being given; - - - - 143

e. The triangle (whose three sides are given) being isosceles or equilateral - - - - - 144

77. To find the area of a trapezoid,

the parallel sides and the distance between them being given; - - - - - - - 144

a. The four sides being given; - - - - 145

b. The altitude, one of the parallel sides, and the two angles adjacent to this side, being given - - - 145

Art. Page
78. From a given triangle to cut off a trapezoid which shall contain a given area - - - - - - 146
79. From a given trapezoid to cut off, by a line parallel to the parallel sides, a part which shall contain a given area - 148
80. In a trapezoid, given the shorter parallel side, the altitude, and the angles adjacent to the other parallel side, to find that side - - - - - - - 151
81. To find the area of a trapezium,
a diagonal and the perpendiculars upon it from the opposite angles being given; - - - - - - 152
a. All the sides and one angle being given; - - 153
b. All the sides and two opposite angles being given; - 154
c. One side, and the two angles subtended at each extremity of the given side by the two sides which are not adjacent to that extremity, being given; - - - 155
d. The diagonals, and the angles which they form with each other, being given - - - - - 156
82. To find the area of an irregular polygon - - - 157
83. To find the area of a regular polygon,
the number of sides of the polygon, and the radius of the circumscribed circle, being given; - - - 159
a. The number of sides, and the radius of the inscribed circle, being given; - - - - - - - 160
b. The number of sides, and the length of a side, being given 161
84. To find the length of the circumference of a circle, the radius or the diameter being given - - - - 162
a. To find the radius or the diameter of a circle, the length of the circumference being given - - - 162
85. Of the three quantities; 1. the length of an arc of a circle; 2. the number of degrees contained in the arc; 3. the radius of the circle; given any two to determine the third 163
86. Given the diameter of a circle to find the area - - 164
a. Given the area of a circle to find the radius - - 165
87. To find the area of a sector of a circle,
the length of the radius, and the number of degrees in the arc of the sector, being given; - - - - 165
a. The radius, and the chord of the arc of the sector, being given; - - - - - - - 166
b. The chord of the arc of the sector, and the perpendicular from the centre of the circle upon the chord, being given - 167
c. The chord of the arc, and the versed sine of half the arc, being given; - - - - - - - 167
d. The chord of the arc of the sector, and the angle at the centre, being given - - - - - - 168
88. To find the area of a segment of a circle; the radius and the number of degrees in the arc of the segment; or the radius and the chord of the segment; or the chord and the altitude of the segment; or the chord and the number of degrees in the arc of the segment; being given - - 168

SECTION IX.

MENSURATION OF SOLIDS.

Art. Page
89. *a....f.* Definitions - - - - - - 171
90. To find the area of the surface of an upright prism - 172
a. To find the area of the surface of an oblique prism - 173
91. To find the area of the surface of an upright pyramid which has a regular polygon for its base,
the perimeter of the base, and the perpendicular from the vertex of the pyramid on a side of the base, being given; 173
a. The number of sides of the base, the length of a side of the base, and the height of the pyramid, being given; - 174
b. The number of sides of the base, the length of a side of the base, and the length of a lateral edge of the pyramid, being given - - - - - - - 174
92. To find the area of the surface of an upright cylinder - 175
a. Area of an equilateral cylinder - - - 176
93. To find the area of the surface of an upright cone - - 177
a. Area of an equilateral cone - - - - 178
94. To find the area of the surface of a frustum of an upright cone,
the radii of the base and top, and the length of the side of the frustum, being given; - - - - 178
a. The radii of the base and top, and the altitude of the frustum, being given - - - - - - 179
b. Relation between the convex surfaces of a cylinder and a frustum of a cone - - - - - - 179
95. To find the area of the surface of a sphere - - - 179
96. To find the area of the convex surface of a segment or zone of a sphere,
the radius of the sphere, and the height of the segment or zone, being given; - - - - - 181
a. To find the area of the convex surface of a segment of a sphere, the radius of the base of the segment and the height being given - - - - - 182
b. To find the area of the whole surface of a spherical segment - - - - - - 182
c. To find the area of the convex surface of a spherical zone, the radii of the base and top, and the height of the zone, being given - - - - - - - 183
d. To find the area of the whole surface of a spherical zone 183
97. To find the volume of a parallelepiped - - - 184
a. The volume of an oblique parallelepiped is equal to the product of the area of a section which is perpendicular to the lateral edges, by the length of one of the lateral edges - - - - - - 185
b, c. Any oblique prism is equivalent to an upright prism whose base is equal to a section of the oblique prism

Art. Page
made at right angles to the lateral edges, and its height equal to the length of a lateral edge of the oblique prism 185
98. To find the volume of a prism - - - - - 185
a. The volume of an oblique prism is equal to the product of the area of a section of the prism made at right angles to the lateral edges, by the length of one of the lateral edges - - - - - - - 186
b. The area of a section of an oblique prism made by a plane perpendicular to the lateral edges of the prism, is obtained by multiplying the area of the base of the prism by the cosine of the inclination of the plane of the base to the plane of the section - - - - 186
99. To find the volume of a pyramid - - - 187
100. To find the volume of a cylinder - - - 188
101. To find the volume of a cone - - - - 188
102. To find the volume of a sphere - - - - 189
a. If d is the diameter of a sphere, $d^3 \times .5236$ expresses the volume of the sphere - - - - 190
103. To find the volume of a frustum of a triangular prism - 190
a. Expression of the volume of a triangular prism which is equal in volume to the frustum - - - 191
b. To find the volume of a wedge - - - 192
104. To find the volume of a frustum of a parallelepiped - 193
a. Expression of the volume of a parallelepiped which is equal in volume to the frustum - - - - 194
b. To find the volume of a frustum of any prism whose base is a regular polygon - - - - - 194
c. To find the volume of a frustum of a cylinder - - 194
105. To find the volume of a parapet or rampart - - 194
106. To find the volume of a prismoidal solid like that which may be conceived to be formed by making an excavation for a road, on sloping ground - - - - 196
107. To find the volume of a frustum of a pyramid or an upright cone - - - - - - - 198
108. To find the volume of a segment of a sphere,
the radius of the sphere, and the height of the segment, being given; - - - - - - - 200
a. The radius of the base of the segment, and the height of the segment, being given; - - - - 201
b. The volume of a segment of a sphere is equal to half the volume of a cylinder which has the same base and height, plus the volume of a sphere which has the height of the segment for its diameter - - - - 202
109. To find the volume of a spherical zone - - - 202

APPENDIX.

1. Of the plane scale - - - - - - 203
2. Of the sector - - - - - - - 206

Art. Page
3. Of the logarithmic scale, or Gunter's scale - - 207
4. Of the protractor - - - - - - 209
5. Of proportional compasses - - - - 209
6. Of Marquois' parallel rulers - - - - 210
7. Of the surveying compass - - - - 212
8. Of the pantagraph - - - - - - 215
9. Solution of a cubic equation by the arithmetic of sines - 216
10. Investigation of logarithmic series for a^x, e^x, e^{-x}, log. $(1+x)$, &c. - - - - - - 218
11. Resolution of triangles, by construction - - - 222
12. Of surveying with a theodolite - - - - 225
13. Of levelling - - - - - - - 229
14. Of the multiplication of duodecimal numbers - - 233

PLANE TRIGONOMETRY

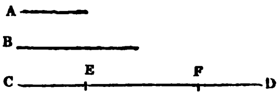
AND

MENSURATION.

SECTION I.

OF THE NUMERICAL VALUES OF STRAIGHT LINES, PARALLELOGRAMS, TRIANGLES, PRISMS, AND PYRAMIDS. — FIRST PRINCIPLES OF THE APPLICATION OF ALGEBRA TO GEOMETRY. — RATIO OF THE DIAMETER TO THE CIRCUMFERENCE OF A CIRCLE.

ARTICLE 1. To find a straight line, which shall be equal in length to two given straight lines, A, B.



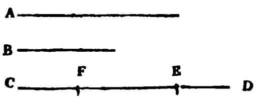
Take a straight line CD, terminated at C, but unlimited towards D: from CD cut off (Euc. I. 3.), CE equal to A; and from ED cut off EF equal to B.

Then, since CE = A, and EF = B
CE + EF = A + B
or CF = A + B.

a. The continuous length of three, four, ... *m* given straight lines can be found by repeating the construction employed in this article.

b. If the three, four, ... *m* given straight lines are all equal, the result obtained by this process is equal to three, four, ... *m*, repetitions of a given straight line.

2. To find a straight line which shall be equal in length to the excess of a given straight line A over another given straight line B.

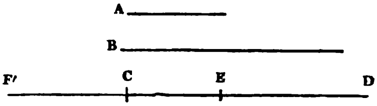


Take a straight line, CD, terminated at C, but unlimited towards D: from CD cut off CE equal to A (Euc. I. 3.); and from EC cut off EF equal to B.

Then, since CE = A, and EF = B
CE − EF = A − B
or CF = A − B.

a. If $B = A$, $EF = CE$, the point F falls upon C, and the difference $CF = 0$.

b. If $B > A$, the length B or EF' cannot be cut off from the line EC or A, unless EC is produced on the other side of C.



In this case the difference $A - B$, expressed by $EC - EF'$ or by $-(EF' - EC)$ is a subtractive or negative quantity. (Alg. art. 3.)

c. The positive lines CE, CF in the preceding figure, are measured from C in the direction CD; and the negative line CF' is measured from the same point C, in the contrary direction. The difference of the directions of two portions of the same line, commencing from the same point, is in this instance expressed by a change in the sign of the line.

d. Since the difference, when A is greater than B, is $+\,CF$,
when A is equal to B, 0,
when A is less than B, $-\,CF$,
it appears that the line CF, in its change of sign, passes through zero.

e. The excess of a given straight line A over two, three, ... given straight lines B, C, D .... can be found by making $CE = A$, and measuring from E, in the contrary direction, $EF = B$; $FG = C$; $GH = D$; &c.

f. If $B = C = D = \ldots$, the line A is diminished by two, three, ... repetitions of the line B.

g. Hence, two unequal straight lines being given, it can be found how often the less is contained in the greater, and whether there be a remainder: if so, that remainder is manifestly shorter than the less line.

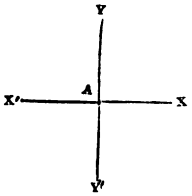
h. By combining the processes of Articles 1. and 2., a straight line may be found which shall be equal to the sum of the lengths of any given straight lines, diminished by the sum of the lengths of any other given straight lines.

3. The negative line CF' obtained as the excess of EF' or B over EC or A, in the case of B greater than A, is situated, with respect to the point C, in a direction contrary to that in which the positive line CE is situated. The following rule for the change of sign of a line, consequent on the change of direction of the line from the right to the left, or from the left to the right of a fixed point, is given by Des Cartes: —

If from a fixed point, or origin, taken in any line straight or curved, distances are measured in contrary directions, the contrariety of direction is accompanied with a corresponding contrariety of sign; so that if lines measured from the point, to the right, are affected with the sign +, then lines measured from the point, to the left, are affected with the sign —.

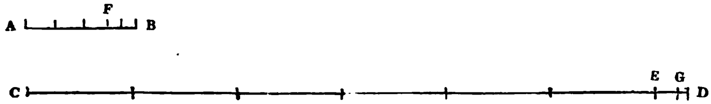
This rule does not admit of demonstration, but is to be considered conventional. A point being taken as an origin, the direction in which lines measured from that point are to be considered positive is arbitrary: this being established, all straight lines measured from the origin, in that direction, must, during the same investigation, be regarded as positive; and those measured from the same origin in the opposite direction must be regarded as negative.

Let X′X, YY′ be two straight lines, which intersect, at right angles, in A: then, by this convention, if lines measured from A in the directions AX, AY, are considered positive, lines measured from A in the directions AX′, AY′, must be considered negative.



4. To find the expression of the numerical value of a given straight line in terms of an arbitrary unit of length.

Let CD be the given straight line, and AB the unit of length.



Find how often AB is contained in CD; and let CD = 6 AB + DE (2. g.).

In the same manner, find how often DE is contained in AB: and let AB = 3DE + BF.

Again, find how often BF is contained in DE: and let DE = BF + DG.

Suppose, lastly, that DG is contained in BF twice, without any remainder.

Then, since BF = 2DG, and DE = BF + DG;

$$\text{DE} = 2\text{DG} + \text{DG} = 3\text{DG}.$$

Next, $\text{AB} = 3\text{DE} + \text{BF} = 3 \times 3\text{DG} + 2\text{DG} = 9\text{DG} + 2\text{DG} = 11\text{DG}.$

Lastly, $\text{CD} = 6\text{AB} + \text{DE} = 6 \times 11\text{DG} + 3\text{DG} = 66\text{DG} + 3\text{DG} = 69\text{DG}.$

$$\therefore \frac{\text{CD}}{\text{AB}} = \frac{69\text{DG}}{11\text{DG}} = \tfrac{69}{11} = 6\tfrac{3}{11}.$$

Consequently, the value of AB being 1, the numerical value of the length CD is $6\frac{3}{11}$.

Making CD = 1, AB $= \frac{11}{69}$.

Making DG = 1, AB = 11, and CD = 69. Therefore AB, CD, being two straight lines whose numerical ratio is required, that ratio is expressed by the fraction $\frac{69}{11}$ or $\frac{11}{69}$.

a. This process, in which the greater line is divided into parts equal to the less, the less into parts equal to the remainder, &c., is like that for finding the greatest common measure of two numbers. (Arith. Art. 107.) DG, the last divisor, is evidently the greatest straight line which is contained an exact number of times in AB, CD: it is therefore the greatest common measure of these lines.

The investigation of the numerical value of a given straight line in terms of an arbitrary unit of length, or the investigation of the numerical ratio of two given straight lines, consequently involves the investigation of the greatest common measure of the given line and linear unit, or of the two given straight lines.

The numbers 11 and 69, which express the ratio of the lines AB and CD, are prime to each other. (Arith. Art. 109.)

In the process of this article, two cases may occur: —

b. After a less or greater number of partial divisions, a remainder may be found which is an exact measure of the preceding remainder. In this case, the two straight lines are said to be commensurable: and the greatest common measure being made 1, the ratio of the proposed lines is expressed by the ratio of two whole numbers; or,

c. It may be impossible, however far the partial divisions are carried, to find a remainder which is an exact measure of the preceding remainder. In this case the two straight lines, having no assignable common measure, are said to be incommensurable with each other. Since the exact value of the ratio of the two straight lines cannot be, in this case, expressed by a fractional number, it becomes necessary to have recourse to an approximation as far extended as possible. In this approximation one of the remainders given by the successive operations (and which may be rendered less than any assigned line, Euc. Prop. A. Proportion) is neglected, and the corresponding quotient is regarded as being complete.

The degree of approximation to the true value of the ratio or of the straight line (if one of the lines compared = 1) is close according as the number of partial operations is considerable, and the remainder neglected is small.

5. To render the investigation of Article 4. more general, let the straight lines, whose numerical expressions are required, be denoted by A, B.

Let B be contained in A, q times with a remainder R.
R . . . B, q' . . . . R'.
R' . . . R, q'' . . . . R''.
R'' . . . R', q''' . . . . R'''.

$$\text{Therefore } A = qB + R \text{ and } \frac{A}{B} = q + \frac{R}{B} \quad . \quad . \quad . \quad 1.$$

$$B = q'R + R', \text{ and } \frac{B}{R} = q' + \frac{R'}{R} \quad . \quad . \quad 2.$$

$$R = q''R' + R'', \text{ and } \frac{R}{R'} = q'' + \frac{R''}{R'} \quad . \quad . \quad 3.$$

$$R' = q'''R'' + R''', \text{ and } \frac{R'}{R''} = q''' + \frac{R'''}{R''} \quad . \quad . \quad 4.$$

$$\text{Now, (eq. 1.) } \frac{A}{B} = q + \frac{R}{B} \text{ and } \frac{R}{B} = 1 \div \frac{B}{R} = 1 \div \left(q' + \frac{R'}{R}\right) = \frac{1}{q' + \frac{R'}{R}} \text{ (Eq. 2.)}$$

$$\therefore \frac{A}{B} = q + \frac{1}{q' + \frac{R'}{R}}$$

$$\text{Again } \frac{R'}{R} = 1 \div \frac{R}{R'}, \text{ and } \frac{R}{R'} = q'' + \frac{R''}{R'} \text{ (Eq. 3.)}$$

$$\therefore \frac{R'}{R} = 1 \div \left(q'' + \frac{R''}{R'}\right) = \frac{1}{q'' + \frac{R''}{R'}},$$

$$\text{And } \frac{A}{B} = q + \cfrac{1}{q' + \cfrac{1}{q'' + \cfrac{R''}{R'}}}$$

Whence the value of the ratio $\frac{A}{B}$ is expressed by the continued fraction,

$$\frac{A}{B} = q + \cfrac{1}{q' + \cfrac{1}{q'' + \cfrac{1}{q''' +}}}, \text{ \&c.}$$

When the result is expressed as a continued fraction, if the fraction consists of a limited number of integrant fractions, the ratio of the two straight lines is expressed by the last convergent: But if the fraction is composed of an unlimited number of integrant fractions, or is indefinitely prolonged, in order to have the exact value of the ratio sought, the continued fraction must be considered under this indefinite form.

The approximation to the exact value will be near according as a greater number of integrant fractions is taken and a greater number of convergents formed from them.

6. Practically, the length of a line is measured by applying continuously along that line a measure equal to the principal unit or some multiple of the principal unit of length, until the remainder is less than this principal unit: applying, in

the same manner, along this remainder, a measure equal to the unit, or some multiple of the unit, which is immediately inferior to the principal unit, and so on.

Following this method, and taking yards, feet, inches, and lines as the principal and inferior units of length,—if a given line contains the principal unit, a yard, 35 times, with a remainder less than 1 yard; if this remainder contains the second unit, a foot, 2 times with a remainder less than 1 foot; if the second remainder contains the third unit, an inch, 8 times with a remainder less than 1 inch; and if the third remainder contains the fourth unit, a line, 4 times, without any remainder, the length of the whole line is 35 yds. 2 ft. 8 in. 4 l.

The measurement, however, of one line by the direct application to it of another line, or by means of compasses, cannot be made with great accuracy. For the detail of different processes, see Art. 60.; Appendix, Art. 1.; and Practical Astronomy and Geodesy, Art. 391.

7. If the numerical values of straight lines are expressed in terms of the same unit of length, the numbers which express these values contain the unit, or some part or parts of the unit of number, as often as the corresponding lines contain the line or the same part or parts of the line assumed as the unit of length (Art. 4.). Whence,

a. The sum or difference of the numerical values of any straight lines has the same relation to 1 as the sum or difference of the lines has to the unit of length. Wherefore the sums and differences of straight lines can be expressed by numbers.

b. The ratio of two straight lines is a number which, unless one of the lines is a measure of the other, is expressed by an irreducible fraction (Art. 4. *a.*). The ratio of the numerical values of the lines is identical with this number, whether integral or fractional. Consequently ratios and proportions among straight lines can be expressed by numbers.

c. Triangles and parallelograms have to one another the same ratio as the products of the numerical values of their bases and altitudes have to one another. For it is proved, Euc. VI. 1. Cor. 3. that triangles and parallelograms are to one another in a ratio compounded of the ratios of their bases and altitudes: that is, if triangles T, T', and parallelograms P, P', have bases B, B', and altitudes A, A' respectively,—

$$\frac{\mathrm{T}}{\mathrm{T}'}=\frac{\mathrm{P}}{\mathrm{P}'}=\frac{\mathrm{B}}{\mathrm{B}'}\times\frac{\mathrm{A}}{\mathrm{A}'}.$$

Now the expressions $\frac{\text{B}}{\text{B}'}$, $\frac{\text{A}}{\text{A}'}$, are numbers identical with the ratios of the numerical values B, B′; A, A′ (Art. 7. *b.*). Whence, replacing these lines by their numerical values b, b', a, a':

$$\frac{\text{T}}{\text{T}'}=\frac{\text{P}}{\text{P}'}=\frac{b}{b'}\times\frac{a}{a'}=\frac{ba}{b'a'}.$$

$$\text{or, T : T}' :: \text{P : P}' :: ba : b'a'.$$

d. If $\text{B} = \text{A}$ and $\text{B}' = \text{A}'$, $b = a$ and $b' = a'$: therefore,

$$\text{T : T}' :: \text{P : P}' :: b^2 : b'^2 :: a^2 : a'^2.$$

But when $\text{B} = \text{A}$ and $\text{B}' = \text{A}'$, P, P′ are equivalent to the squares of B, B′. Therefore squares and consequently (Euc. VI. 20. Cor.) all similar rectilineal figures, are to each other as the squares of the numerical values of their sides. They are also to one another as the squares of the numerical values of their altitudes.

e. Let B and A, the base and altitude of P, be any lines, and let B′, A′, the base and altitude of P′, be each equal to the unit of length. Then since B, A, are any lines, b, a, are any numbers: and since $\text{B}' = \text{A}' =$ the unit of length, $\text{B}' \times \text{A}' = \text{B}'^2 =$ the square described on the unit of length; also $b' = a' = 1$, and $b' \times a' = 1 \times 1 =$ the unit of number. Consequently the proportion $\text{B} \times \text{A}$ or P : $\text{B}' \times \text{A}'$ or P′ :: $ba : b'a'$, becomes

$$\text{B} \times \text{A or P : B}'^2 :: ba : 1.$$

Assuming the square of the unit of length as the unit of surface, it follows from this proportion, that the rectangle under the base and altitude of a parallelogram contains the unit of surface as often as the product of the numerical values of the base and altitude contains 1. Hence the *area* (or the measure of the surface expressed numerically) of a parallelogram is obtained by forming the product of the numerical values of the base and altitude of the parallelogram.

f. Since a triangle is equivalent to the half of a parallelogram which has the same base and altitude (Euc. I. 41.), it follows that the area of a triangle is obtained by forming the product of the numerical values of the base and altitude of the triangle and taking the half of this product.

g. It is proved (Prisms, &c. Prop. 7. Cor.) that parallelepipeds are to one another as the products of any numbers which are proportional to the areas of their bases and to their altitudes. Let therefore P, P′ be two parallelepipeds, and let L, B, H, denote the length, breadth, and height of P,

L′, B′, H′, the length, breadth, and height of P′.

Also let l, b, h, denote the numerical values of L, B, H, and l', b', h', the numerical values of L′, B′, H′, respectively.

Then $\text{LB} : \text{L}'\text{B}' :: lb : l'b'$ (Art. 7. *c.*). Whence lb, $l'b'$, are numbers proportional to the bases of the parallelepipeds P, P′. Also, h, h', the numerical values of H, H′, are proportional to H, H′ (Art. 4.).

$$\text{Consequently, } \text{P} : \text{P}' :: \text{LBH} : \text{L}'\text{B}'\text{H}' :: lbh : l'b'h'.$$

Let L, B, H, the length, breadth, and height of P, be any lines; and let L′, B′, H′, the length, breadth, and height of P′, be each equal to the unit of length. Then since L, B, H, are any lines, l, b, h, are any numbers: and since $\text{L}' = \text{B}' = \text{H}' =$ the unit of length, $\text{L}'\text{B}'\text{H}' = \text{L}'^3 =$ the cube described on the unit of length: also $l' = b' = h' = 1$, and $l'b'h' = 1 \times 1 \times 1 =$ the unit of number;

$$\text{wherefore, } \text{P} : \text{P}' :: \text{LBH} : \text{L}'^3 :: lbh : 1.$$

Assuming the cube described on L′, the unit of length, as the unit of solid measure, it follows from this investigation, that the parallelepiped whose length, breadth, and height are L, B, H, is to the cube of the unit of length, as l, b, h, the product of the numerical values of L, B, H, is to 1. Hence the *volume* (or the measure of the solid content, expressed numerically) of a parallelepiped, is obtained by forming the continual product of the numerical values of the length, breadth, and height of the parallelepiped.

h. Since a prism is equivalent to a parallelepiped (Prisms, &c., Prop. 9.), and a pyramid is one third of a parallelepiped (Prisms, &c., Prop. 15. Cor. 1.), which has an equivalent base and an equal altitude, it follows that the property established (Art. 7. *g.*) in the case of parallelepipeds is also true in the cases of prisms and pyramids.

8. The relations of the sums, differences, and ratios of straight lines being expressible by corresponding relations among the numerical values of these straight lines; and the relations of plane rectilineal surfaces and of solids bounded by plane surfaces, by corresponding relations of the products of the numerical values of the straight lines containing these surfaces and solids, it is evident that the relations among lines and figures, which are contemplated in Elementary Geometry, are reducible to corresponding relations among the numerical values of these lines and figures.

Now numbers admit of representation by letters: whence if the data of a geometrical proposition are expressed in terms of their numerical values, and these values are denoted by letters; and if the unknown quantity or quantities involved in the proposition are also denoted by letters, and the relations of the quantities, known and unknown, are expressed by algebraic symbols, the geometrical proposition is translated into the language of algebra.

It is, in general, necessary or useful to make some construction connecting the given and the unknown quantities, so as to facilitate the translation of the conditions of the proposition into algebraic language, and the deduction of the required result from the algebraic representatives of the geometrical quantities. The following directions have been found useful for this purpose: —

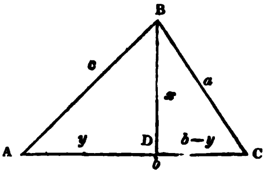
1st. Draw all the lines, known and unknown, mentioned in the proposition.

2d. Draw lines connecting the given and unknown quantities; and when convenient let the connecting lines be parallel or perpendicular to the lines of the figure: because the relations of proportionals among the parallel lines, and of equations between the squares of the sides of the right-angled triangles, are given by these arrangements.

3d. Consider the connecting lines as auxiliary unknown quantities; and form equations connecting together the lines of the figure. When the problem is determinate, there will be thus obtained a number of equations equal to that of the unknown lines, and the solution of the geometrical proposition will be reduced to the resolution of these equations.

a. As illustrations of the method to be followed in applying the processes of algebra to the resolution of determinate problems in geometry, 1st, let the numerical values of the three sides of a triangle be given, and let it be required to find the area of the triangle.

The triangle being A B C, let the numerical values of the sides B C, A C, A B, be denoted by a, b, c, respectively. From B, draw B D perpendicular to A C; denote B D by x, A D by y, and therefore D C by $b-y$.



Since the area of the triangle A B C $= \frac{\text{AC} \times \text{BD}}{2} = \frac{bx}{2}$ (Art. 7. *f.*), it is necessary to express B D or x in terms of the given quantities, a, b, c, or of some of these quantities. Now since the triangles A B D, D B C, are right-angled,

$BD^2 = BC^2 - DC^2$ or $x^2 = a^2 - (b-y)^2$ or $x^2 = a^2 - b^2 + 2by - y^2$ . . . 1

$BD^2 = AB^2 - AD^2$ . . . or $x^2 = c^2 - y^2$ . . . 2

Subtracting equation 2 from equation 1

$$0 = a^2 - b^2 - c^2 + 2\,by$$

Transposing and dividing, $y = \frac{b^2 + c^2 - a^2}{2\,b}$.

Substituting this value of y in equation 2, the value of x^2 expressed in terms of a, b, c, is $x^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4\,b^2}$.

$$\therefore x = \sqrt{c^2 - \frac{(b^2 + c^2 - a^2)^2}{4\,b^2}}.$$

The area represented by $\frac{b\,x}{2}$ can be found from this expression: for since

$$x^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4\,b^2}$$

$$\therefore 4\,b^2x^2 = 4\,b^2c^2 - (b^2 + c^2 - a^2)^2$$

$$\text{or, } 4\,b^2x^2 = (2\,bc)^2 - (b^2 + c^2 - a^2)^2.$$

But because $(a + b)\,(a - b) = a^2 - b^2$, the second member of this equation, which is composed of the difference of two squares, can be resolved into the product of the factors

$$2\,b\,c + b^2 + c^2 - a^2,\ 2\,b\,c - b^2 - c^2 + a^2$$

$$\text{or, } b^2 + 2\,b\,c + c^2 - a^2,\ a^2 - b^2 + 2\,b\,c - c^2$$

$$\text{or, } (b + c)^2 - a^2,\ a^2 - (b - c)^2.$$

Again, for the same reason,

$$(b + c)^2 - a^2 = (b + c + a)\,(b + c - a)$$

$$a^2 - (b - c)^2 = (a + b - c)\,(a + c - b).$$

Whence, substituting the products of the sums and differences of the numbers for the differences of the squares of these numbers,

$$4\,b^2x^2 = (b + c + a)\,(b + c - a)\,(a + b - c)\,(a + c - b)$$

or (Arith. Art. 71.)

$$4\,b^2x^2 = (a + b + c)\,(b + c - a)\,(a + c - b)\,(a + b - c).$$

Denoting the sum of the sides, or the perimeter, of the triangle by $2\,s$,

$$a + b + c = 2\,s,$$

$$\text{but } 2\,a = 2\,a$$

$\therefore$ by subtraction, $b + c - a = 2\,s - 2\,a = 2\,(s - a)$

In like manner, $a + c - b = 2\,(s - b)$

$$a + b - c = 2\,(s - c)$$

Consequently, by substitution,

$$4b^2x^2 = 2s \times 2(s-a) \times 2(s-b) \times 2(s-c)$$

$$\text{or, } 4b^2x^2 = 16s(s-a)(s-b)(s-c). \text{ (Arith. Art. 71.)}$$

Extracting the square root,

$$2bx = 4\sqrt{s(s-a)(s-b)(s-c)}.$$

Dividing both members of this result by 4, because the area of the triangle $= \frac{bx}{2}$

$$\frac{bx}{2} = \sqrt{s(s-a)(s-b)(s-c)}.$$

If the angle BAC is obtuse,

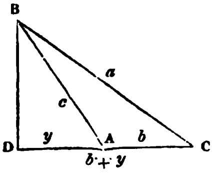
$$x^2 = a^2 - (b+y)^2 = a^2 - b^2 - 2by - y^2$$

$$x^2 = c^2 - y^2$$

$$\therefore\ 0 = a^2 - c^2 - b^2 - 2by$$

$$\text{And, } y = \frac{a^2 - c^2 - b^2}{2b} = \frac{-(b^2 + c^2 - a^2)}{2b}$$

$$\text{Or } -y = \frac{b^2 + c^2 - a^2}{2b}$$



This negative value of y corresponds with the direction, AD, which is opposite to the direction AD, in the case of A an acute angle.

Since $(-y)^2 = +y^2$, this difference of sign causes no change in the value of $x^2 = c^2 - y^2$, or in that of the area of the triangle deduced from the expression $\frac{bx}{2}$.

Since the sum of any two sides of a triangle is greater than the third side, (Euc. I. 20.) half the sum of any two sides is greater than half the third side: therefore, half the sum of any two sides + half the third side is greater than the third side; or the semi-perimeter of any triangle is greater than any side of that triangle. Consequently all the arithmetical subtractions $s-a$, $s-b$, $s-c$, can be made, and the product $s(s-a)(s-b)(s-c)$ is such, (being positive, because the product of factors which are positive) that its square root can be, in all cases, extracted either exactly or with any required degree of approximation.

2nd. Let it be required to find the ratio of the circumference to the diameter of a circle.

The circumference of a circle is greater than the perimeter of any regular polygon inscribed in the circle, and less than the perimeter of any regular polygon described about it. But regular polygons of the same number of sides can be inscribed in and described about a circle, the perimeters of which polygons differ from each other, and from the circumference of the circle included between them by lines which are less than any assigned lines (Circle, prop. 6.). This is accomplished by inscribing in and describing about the circle regular polygons of a certain number of sides; then polygons of twice the number of sides, polygons of four times the number of sides, and so on.

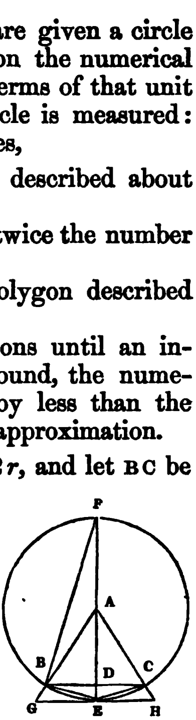
To resolve the problem, therefore, there are given a circle of known diameter and an inscribed polygon the numerical value of a side (s) of which is expressed in terms of that unit of length by which the diameter of the circle is measured: and it is required to find the numerical values,

1st. Of the side (S) of the similar polygon described about the circle.

2d. Of the side (s') of a regular polygon of twice the number of sides inscribed in the circle.

3d. Of the side (S$'$) of the corresponding polygon described about the circle.

4th. And to continue this series of operations until an inscribed and circumscribed polygon are found, the numerical values of whose perimeters differ by less than the number which is to form the limit of the approximation.



Let BECF be a circle whose diameter is $2r$, and let BC be the side of a regular polygon inscribed in this circle. From the centre A, draw AD perpendicular to BC, (Euc. I. 12.) produce AD to the circumference in E, F; through E, draw GH touching the circle in E (Euc. III. 17.), through A, B, draw AG; through A, C, draw AH; and join BE, BF.

By this construction GH is the side of a polygon described about the circle similar to the inscribed polygon whose side is BC (Circle, prop. 2.); and BE is the side of a polygon having twice as many sides as the polygon BC.

Assuming, then, that BC or s is given, it is required to assign the values of GH or S and BE or s', in terms of $\text{AE} = r$ and $\text{BC} = s$.

Since BC is parallel to GH, AD : AE :: BC : GH (Euc. VI. 4. and V. 1.).

$$\text{or,}\quad \text{AD} : r :: s : \text{S} \quad . \; . \; . \; . \; 1$$

Also, $\text{AD} = \sqrt{\text{AB}^2 - \text{BD}^2} = \sqrt{\text{AB}^2 - \frac{1}{4}\text{BC}^2} = \sqrt{r^2 - \frac{1}{4}s^2}$ (Euc. I. 47.).

Substituting this value of AD in eq, 1. $\sqrt{r^2 - \frac{1}{4}s^2} : r : s : \text{S}$;

$$\therefore \; \text{S} = \frac{rs}{\sqrt{r^2 - \frac{1}{4}s^2}}$$

Again, since FBE is a right angle, (Euc. III. 31.) and BD is perpendicular to FE,

$$BE^2 = FE \times ED. \quad \text{(Euc. VI. 8. and 17.)}$$

Now, $BE = s' : FE = 2AB = 2r$; $BD = AB - AD = r - \sqrt{r^2 - \frac{1}{4}s^2}$.

$$\therefore\ s'^2 = 2r\left(r - \sqrt{r^2 - \tfrac{1}{4}s^2}\right).$$

$$\text{And, } s' = \sqrt{2r\left(r - \sqrt{r^2 - \tfrac{1}{4}s^2}\right)}.$$

Denoting the value of the perpendicular AD, or $\sqrt{r^2 - \frac{1}{4}s^2}$, by p, the formulæ which give S, s' in terms of s and r are

$$\left.\begin{array}{l} p = \sqrt{r^2 - \frac{1}{4}s^2}; \\ S = \dfrac{rs}{p}; \\ s' = \sqrt{2r(r-p)}. \end{array}\right\} . \text{ A.}$$

It is evident that, in the same manner, from r, s', can be found

$$p' = \sqrt{r^2 - \tfrac{1}{4}s'^2};$$

$$S' = \frac{rs'}{p'},$$

$$s'' = \sqrt{2r(r-p')},$$

that similarly p'' can be found from r, s''; S'' from r, s'', p''; and that the process can be indefinitely repeated.

$$\text{If } r = 1,\ p = \sqrt{1 - \tfrac{1}{4}s^2};$$

$$S = \frac{s}{p};$$

$$s' = \sqrt{2(1-p)}.$$

To apply formulæ A to the determination of the approximate ratio of the circumference to the diameter of a circle; let BC be the side of a regular hexagon inscribed in a circle whose radius is 1:

$$\text{Then, } s = 1. \quad \text{(Euc. IV. 15. Cor.)}$$

$$p = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = .8660254037845$$

$$S = \frac{1}{.8660254037845} = 1.15470054$$

Since $s = 1$, $6s = 6$ = perimeter of inscribed hexagon;

And since $S = 1.15470054$, $6S = 6.92820323$ = perimeter of circumscribed hexagon.

The circumference of the circle whose diameter is 2 is therefore greater than 6, and less than 6.92820323.

Next, to find the perimeters of the inscribed and circumscribed dodecagons:

$$s' = \sqrt{2(1-p)} = \sqrt{2(1-0.8660254037845)} = \sqrt{.267949192431} = .517638090205;$$

$$p' = \sqrt{1-\tfrac{1}{4}s'^2} = \sqrt{1-\frac{.267949192431}{4}} = \sqrt{.933012701892} = .965925826289;$$

$$S' = \frac{s'}{p'} = \frac{.517638090205}{.965925826289} = .535898388.$$

Hence the perimeter of the inscribed dodecagon = .517638090205 × 12 = 6.21165708246;
And the perimeter of the circumscribed dodecagon = .535898388 × 12 = 6.430780656.

The circumference of the circle whose diameter is 2, is consequently greater than 6.21165708246, and less than 6.430780656.

In the same manner, from the values of r, s', p', are found

$$s'' = \sqrt{2(1-p')} = .26105238444,$$

$$p'' = \sqrt{1-\tfrac{1}{4}s''^2} = .991444861374,$$

$$S'' = \frac{s''}{p''} = .2633049957.$$

s'', S'' are respectively the numerical values of a side of the inscribed and circumscribed regular polygon of 24 sides.

By continuing this series of operations, the numerical values of the perimeters of regular polygons of 48, 96, 192, 384, $2^n \times 6$ sides, inscribed in and described about the circle whose diameter is 2, are found. The results are contained in the following table: —

Number of Sides of Polygon.	Perimeter of inscribed Polygon.	Perimeter of circumscribed Polygon.
6	6.0000000	6.9282032
12	6.2116571	6.4307806
24	6.2652572	6.3193199
48	6.2787004	6.2921724
96	6.2820639	6.2854292
192	6.2829049	6.2837461
384	6.2831152	6.2833260
768	6.2831678	6.2832203
1536	6.2831809	6.2831941
3072	6.2831842	6.2831875
6144	6.2831850	6.2831858
12288	6.2831852	6.2831854

From these results it appears that as the number of sides of the polygons is increased, the numerical values of the perimeters of the inscribed polygons are augmented, and the numerical values of the perimeters of the circumscribed polygons diminished; so that the ratio of the perimeters constantly tends to a ratio of equality; and each of the perimeters to an equality with the circumference which forms the limit between them. Hence the figures which are common to the numerical values of both polygons must necessarily belong to the numerical value of the circumference of the circle. The circumference of a circle whose diameter is 2 is consequently 6.283185, or (taking the arithmetical mean between the perimeters of the inscribed and circumscribed polygons of 12288 sides), 6.2831853.

The circumferences of circles being to each other in the same proportion as their diameters (Circle, Prop. 10.), the circumference of a circle whose diameter is 1 is ½ of 6.2831853 or 3.1415926.

This result is an approximation to the exact value, which cannot be expressed by any number of figures. The calculation has been carried to 150 decimal figures. Of these, the first twenty-five are

$$3.1415926535897932384626433.$$

b. Let C, c, denote the circumferences of two circles, valued as straight lines; R, r, their radii; and $2R$, $2r$, their diameters respectively. Then

$$C : c :: 2R : 2r$$

$$\text{And } \frac{C}{2R} = \frac{c}{2R}.$$

Hence every circumference contains its diameter the same number of times. This constant number, equal to the ratio $\frac{c}{2r}$ is denoted by π.

$$\text{Since } \frac{c}{2r} = \pi,\ c = 2\pi r.$$

Making $r = 1$, $c = 2\pi$. The circumference of a circle whose radius is 1 may consequently be denoted by 2π.

The numerical value of π is 3.1415926 +.

c. Denoting by x the diameter of the circle whose circumference is 1,

circumf. 3.1415926 : circumf. 1 :: diameter 1 : diam. x:

$$\therefore\ x = \frac{1}{3.1415926} = .3183099.$$

Whence .3183099 expresses the diameter of that circle whose circumference is 1.

d. The preceding are instances of the resolution of problems by the application of algebra to geometry. If, as in each of these problems, the result sought is numerical, the process ends with the deduction of the equation or equations which express the unknown in terms of the given quantities. But if geometrical results are required, it becomes necessary to construct the geometrical equivalents of the numerical values expressed by the final equations.

Without entering into the details of this subject it may be sufficient to explain that a straight line is constructed by taking the same multiple of the unit of length as the numerical value of the line is of 1: that lines being, in this manner, constructed from their numerical values, their sums and differences are found by Articles 1. and 2.; third, fourth, and mean proportionals by Euc. VI. 11, 12, 13.; the geometrical values of expressions such as $x = \sqrt{a^2 + b^2}$; $x = \sqrt{a^2 - b^2}$ or $x^2 = a^2 + b^2$; $x^2 = a^2 - b^2$, by making a, b, the sides about the right angle, and x, the hypothenuse of a right-angled triangle, in the first instance; b, one of the sides about the right angle, a, the hypothenuse, and x, the other side about the right angle, in the second.

9. Among the geometrical propositions from which equations are commonly derived are Euc. El. Prop. 47. b. I., and Props. 12. and 13. b. II., which give the relations between the squares of the sides of triangles: the propositions respecting the equivalence of the rectangles and squares of different parts of the same line in book II.; the propositions which establish the equivalence of the rectangles contained by the segments of two lines intercepted between the same point and the circumference of a circle, being propositions 35. and 36. book III.; the propositions respecting the proportionality of the sides of equiangular triangles, and the ratios of surfaces in book VI., and the propositions respecting the ratios of the volumes of parallelepipeds, prisms, pyramids, cones, cylinders, and spheres in solid geometry.

a. In all these propositions, lines being compared with lines, surfaces with surfaces or the squares of lines, and volumes with volumes or the cubes of lines, and a line being denoted by an expression of one dimension, as a, b; a surface, by an expression of two dimensions, as ab, a^2; a volume, by an expression of three dimensions as abc, a^2b, a^3, it follows that any equation derived from a geometrical proposition must be composed of terms which are all homogeneous. The algebraic processes employed to disengage the representatives of the unknown quantities, and to express them in terms of the known quantities, cannot alter the degree of one term of any of the equations without altering similarly that of all the other terms: whence the result must be also composed of homogeneous terms.

b. There is one case of exception,—when the unit of geometrical magnitude, whether line, surface, or volume, enters into any of the terms of the equation. This 1, if combined by multiplication or division with other expressions of number, whether particular or general, disappears, and the homogeneity of the expressions is destroyed. In this case it becomes necessary to multiply the terms which are of dimensions inferior to the others by such powers of the measuring unit as render the terms homogeneous. For example, the expression $x = \frac{a^3 - 2a + 3bc}{a - 2b^2 + 1}$, is not homogeneous, one term of the numerator being of the third degree, and one term of the denominator, of the second degree; it therefore becomes necessary to reduce the other terms of the numerator and denominator to the third and second degree, respectively. Denoting the measuring unit by *m*,

the result is $x = \frac{a^3 - 2am^2 + 3bcm}{am - 2b^2 + m^2}$.

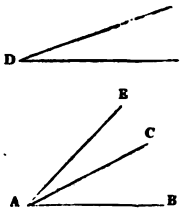
SECTION II.

OF THE NUMERICAL VALUES OF ANGLES AND CIRCULAR ARCS.

10. To find an angle which shall be equal to the sum of two given angles.

Let BAC and D be the given angles.

At the point A (which is the vertex of the angle BAC), make the angle CAE equal to the given angle D (Euc. I. 23.).



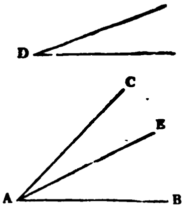
Then $BAE = BAC + CAE = BAC + D$ = the sum of the given angles.

a. It is evident that by repetitions of this construction, an angle may be made equal to the sum of any given angles; or (if the angles whose sum is required are all equal to a given angle BAC) to any multiple of a given angle.

11. To find an angle which shall be equal to the difference of two given angles.

Let BAC and D be the given angles.

At the point A, in the straight line CA, make the angle CAE equal to D.



Then $BAE = BAC - CAE = BAC - D$ = the difference of the given angles.

a. By repetitions of this construction, the difference between one angle and several smaller angles may be found: also the number of angles, each equal to the less of two given angles, which can be taken from the greater until either no remainder is left or a remainder smaller than the less of the two given angles.

b. In taking the angle D from the angle BAC, if D is greater than BAC, the difference, $BAC - D$, is negative.

12. By means of the last article and a process exactly like that of which the details are given in Article 4., the greatest common measure of two angles can be determined either exactly when the angles are commensurable, or with any required accuracy of approximation when the angles are not commensurable. The numerical values of these angles can

consequently be expressed in terms of any angle which may be assumed as the unit of angular magnitude.

a. The right angle being divided into 90 equal parts, one of these parts (which is named *degree,* and written 1°) is assumed as the principal unit of angular magnitude; and, like other measuring units, it is arbitrary. To measure angles less than the 90th part of the right angle, the degree is divided into 60 equal parts, of which one is named *minute,* and written $1'$ or 1^{m}. The minute is also divided into 60 equal parts, which are named *seconds,* and written $1''$ or 1^{s}. Angles less than $1''$ are generally expressed as decimal parts of a second. An angle of thirty-five degrees, nineteen minutes, twenty-eight seconds, and six-tenths of a second is expressed in figures as follows:—$35^{\circ}\ 19'\ 28''.6$.

The numbers by which the value of an angle is thus expressed are termed sexagesimal numbers, because 60 of each of the lower denominations make 1 of the next higher denomination.

b. Another division of the right angle into 100 equal parts, or *grades,* of the grade into 100 equal parts or *minutes,* and of the minute into 100 equal parts or *seconds,* is employed in many French mathematical works. This division is termed the centesimal, because 100 of each of the lower denominations make 1 of the next higher denomination.

Denoting centesimal degrees by $^{gr.}$; minutes by $^{\backprime}$, and seconds by $^{\backprime\backprime}$;

$$90^{\circ} = 100^{gr.}$$

$$\therefore\ 1^{\circ} = \left(\frac{10}{9}\right)^{gr.} \text{ and } 1^{gr.} = \left(\frac{9}{10}\right)^{\circ}$$

$$(90 \times 60)' = (100 \times 100)^{\backprime}$$

$$\text{or } 54' = 100^{\backprime} \text{ or } 27' = 50^{\backprime}$$

$$\therefore 1' = \left(\frac{50}{27}\right)^{\backprime} \text{ and } 1^{\backprime} = \left(\frac{27}{50}\right)'$$

$$(90 \times 60 \times 60)'' = (100 \times 100 \times 100)^{\backprime\backprime}$$

$$\text{or } (9 \times 3 \times 3)'' = (10 \times 5 \times 5)^{\backprime\backprime}$$

$$\text{or } 81'' = 250^{\backprime\backprime}$$

$$\therefore\ 1'' = \left(\frac{250}{81}\right)^{\backprime\backprime} \text{ and } 1^{\backprime\backprime} = \left(\frac{81}{250}\right)''$$

Hence, to convert sexagesimal into centesimal degrees, minutes, &c.

Multiply the sexagesimal degrees by $\frac{10}{9}$,
the minutes by . . . . $\frac{50}{27}$,
the seconds by . . . . $\frac{250}{81}$,

and add the several results into one sum (attending to the rule for the addition of compound numbers): this sum is the equivalent number expressed in terms of the centesimal division.

Similarly, to convert centesimal into sexagesimal degrees, minutes, &c.

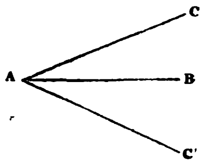
Multiply the centesimal degrees by $\frac{9}{10}$,
the minutes by . . . . $\frac{27}{50}$,
the seconds by . . . . $\frac{81}{250}$,

and add the results into one sum. This sum is the equivalent expression, in terms of the sexagesimal division.

Otherwise, centesimal degrees, &c., may be converted into sexagesimal degrees and decimal parts of a degree by subtracting $\frac{1}{10}$: and sexagesimal degrees, minutes, &c., may be converted into centesimal degrees, &c., by reducing the minutes, &c., to the decimal of a degree, annexing this decimal to the number expressing the degrees, and increasing the number thus obtained by $\frac{1}{9}$th part of itself.

13. If a straight line A B, is made to turn round one extremity A, it traces angles of every magnitude from 0° to four right angles, or 360°. Having revolved through 360°, it resumes its original position, from which it may again be turned round and made to trace continually increasing angles.

In the Elements of Euclid, the greatest angle which can be contained by two straight lines is considered to be less than two right angles, or 180°; but in Trigonometry, the magnitude of an angle is not restricted to either a half or a whole revolution, but is considered to be unlimited.

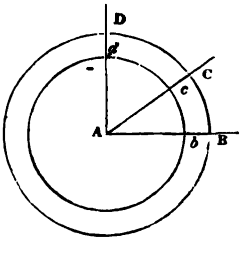


a. If the angles traced by a line revolving from the position A B, in the direction A C are assumed to be positive, then the angles traced by a line revolving from A B in the direction A C′, by a convention like that of Article 3, are considered negative.

14. If a point is taken in the revolving line A B, the point describes an arc of a circle.

The right angle B A D, intercepts a quadrant, or fourth part of the whole circumference of this circle.

If straight lines are drawn dividing the right angle into 90 equal parts, these straight lines divide the quadrant also into 90 equal parts (Euc. VI. 33.): and other straight lines, drawn so as to divide the degree into 60 minutes, and the minute into 60 seconds, divide the 90th part of the quadrant into 60 equal parts, and one of these again into 60 equal parts. Corresponding with the names given to the parts of the right angle, the 90th part of the quadrant is termed degree, the 60th part of the degree, minute, and the 60th part of the minute, second.



a. Hence the measure of an angle being given in degrees, &c., these degrees, &c., are also the measure of the intercepted arc of any circle whose centre is at the vertex of the angle. And conversely, the measure of an arc being given in degrees, &c., these express the measure of the angle at the centre.

b. Wherever the point B is taken in the revolving line, and consequently whatever the length of the radius with which the circle is described, the straight lines which divide the right angle into 90 equal angles divide also the quadrant into 90 equal arcs.

Let B, b be two points taken in the revolving line, and BCD, bcd, the circumferences of the circles described by the radii AB, Ab, respectively. Then, denoting the circumferences BCD, bcd, by C, c, respectively,

$$\mathrm{C} : c :: \mathrm{R} : r \text{ (Circle, Prop. 10.)}$$

$$\therefore \frac{\mathrm{C}}{360^\circ} : \frac{c}{360^\circ} :: \mathrm{R} : r. \text{ (Propor. 1.)}$$

Consequently the length of the arc of one degree in circumference C, exceeds the length of the arc of one degree in circumference c, in the same proportion as the radius R exceeds the radius r.

c. Since the circumference of the circle whose radius is 1, is denoted by 2π (Art. 8. *c.*), and the circumferences of circles are proportional to their radii (Circle, Prop. 10.);

radius 1 : radius AB :: 2π : $2\pi \times \mathrm{AB}$ = the circumference whose radius is AB.

And since, in the same circle, angles are proportional to the arcs on which they stand (Euc. VI. 33.);

$$\text{angle BAC} : 4 \text{ right angles} :: \text{arc BC} : 2\pi \times \mathrm{AB}$$

$$\therefore \text{angle BAC} = \frac{4 \text{ right angles} \times \text{arc BC}}{2\pi \times \mathrm{AB}} = \frac{4 \text{ right angles}}{2\pi} \times \frac{\text{arc BC}}{\mathrm{AB}}.$$

Now $\frac{4 \text{ right angles}}{2\pi}$ is a constant: therefore, $\frac{\text{arc BC}}{\mathrm{AB}}$ or $\frac{\text{arc BC}}{\text{rad.}}$ varies directly as the angle BAC (Alg. Art. 182.). Hence $\frac{\text{arc BC}}{\text{rad.}}$ is a measure of the angle BAC.

d. In order that the ratio $\frac{\text{arc BC}}{\mathrm{AB}}$ or $\frac{\text{arc BC}}{\text{rad.}}$ may have a numerical value, it is necessary that either the value of the arc BC be assigned in terms of a straight line, or that the value of the radius AB be assigned in terms of an arc of the circle whose radius is AB.

Now if the number of degrees in the arc BC is expressed by the symbol m°, and the length of the whole circumference by $2\pi \times \mathrm{AB}$, the numerical value of the arc BC, expressed as a straight line, is given by the proportion,

$$360^\circ : m^\circ :: 2\pi \times \mathrm{AB} : \text{arc BC} = 2\pi . \mathrm{AB} \times \frac{m^0}{360^0}.$$

e. Hence if the radius r of a circle, and an angle m^0 at its centre, are given to find the length l of the corresponding arc of the circle,

$$l = \frac{2\,\pi\,r\,m^0}{360^0}.$$

If the length l of the arc, and the angle m^0 at its centre, are given to find r,

$$r = \frac{l \times 360^0}{2\,\pi\,m^0}.$$

And if the length l of the arc, and the radius r, are given to find the angle m^0 at the centre,

$$m^0 = \frac{l \times 360^0}{2\,\pi\,r}.$$

f. To find the number of degrees subtended by the radius, supposing it curved so as to coincide with a part of the circumference of the circle. The length of the radius being 1, that of the semicircumference is 3.1415926, and the number of degrees subtended by the semicircumference is 180°:

$$\text{therefore, } 3.1415926 : 1 :: 180^\circ : 180^\circ \times \frac{10000000}{31415926} = 57^\circ.29577 = 57^\circ\,17'\,44''.8.$$

The radius of every circle being contained the same number of times in the circumference, it follows that, in every circle, an arc of $57^\circ 17' 44''.8$ or $206264''.8$ is equal in length to its radius. Whence, also, in every circle, the arc of $1''$ is equal to the $\frac{1}{206264.8}$th (or nearly the $\frac{1}{206265}$th) part of the radius.

g. The expression $\frac{\text{arc}}{\text{rad.}}$, being a measure of the magnitude of an angle, the equality, angle $= \frac{\text{arc}}{\text{rad.}}$, may be assumed: or, denoting the angle by ϕ, the arc by a, and the radius by r, $\phi = \frac{a}{r}$.

If in this expression the arc a is equal to the radius, $\frac{a}{r} = \frac{r}{r} = 1$, or the measure of ϕ is 1.

The angle which is subtended by an arc whose length is equal to the radius is, hence, the unit of measure when this method of expressing the magnitudes of angles is employed.

It has been found that in every circle an arc of $57^\circ.29577$ is equal in length to the radius. The numerical value of an angle being expressed in terms of this circular measure, the unit is consequently 57.29577 times as great as the arbitrary unit 1°.

Hence, to convert the numerical value of an angle, expressed in degrees, &c., into an equivalent number expressed in circular measure, it is necessary to divide the number of degrees, &c., by 57.29577: and to convert the numerical value of an angle expressed in circular measure into an equivalent number expressed in degrees, &c., it is necessary to multiply the numerical value in the circular measure by 57.29577.

For methods of measuring a given angle, and constructing an angle of a given number of degrees, see Appendix, Articles 1. and 4.

SECTION III.

TRIGONOMETRICAL FORMULÆ.

15. THE difference between an angle and a right angle is named the complement, and the difference between an angle and two right angles, the supplement of that angle.

In like manner, the complement of an arc is the difference between that arc and a quadrant; and the supplement, the difference between that arc and a semicircumference.

The complement of an angle or arc of 58° 42′ is 90° — 58° 42′ = 31° 18′
The complement of an angle or arc of 96° 25′ is 90° — 96° 25′ = — 6° 25′
The supplement of an angle or arc of 130° 31′ is 180° — 130° 31′ = 49° 29′
The supplement of an angle or arc of 219° 12′ is 180° — 219° 12′ = — 39° 12′.

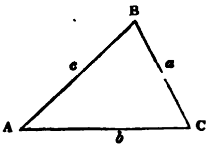
Hence the complement of an angle or arc less than 90°, and the supplement of an angle or arc less than 180°, are positive.

The complement of an angle or arc greater than 90°, and the supplement of an angle or arc greater than 180° are negative.

Hence also, in right-angled triangles, each of the acute angles is the complement of the other acute angle; and in every triangle, each of the angles is the supplement of the other two angles.

16. In the following Articles, the numerical values of the angles of a triangle are denoted by the letters A, B, C; and the numerical values of the sides which are opposite to these angles, by the letters *a*, *b*, *c*, respectively.

The ratio of the numerical values of the sides BC, AC, is expressed by the fraction $\frac{a}{b}$ or $\frac{b}{a}$.



The number of different ratios which can be formed with the three sides of a triangle, taken two and two, is six; (Alg. Art. 136.), viz. $\frac{a}{b}, \frac{b}{a}, \frac{a}{c}, \frac{c}{a}, \frac{b}{c}, \frac{c}{b}$. Like other numerical fractions, these expressions may be reduced to the lowest terms.

a. If the ratio $\frac{a}{b}$, of two sides of any triangle and the numerical value of B C, the side homologous to a in a similar triangle, are given, the numerical value of A C, the side which is homologous to b, can be determined. For, denoting the value of A C by x,

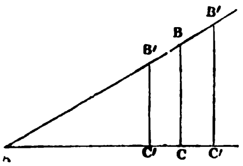
$$\frac{a}{b}=\frac{\text{BC}}{x} \therefore x=\frac{b.\text{BC}}{a}.$$

b. If numbers proportional to any two sides of a right-angled triangle are given, the number which has the same proportion to the third side of that triangle is given by one of the formulæ,

$$a=\sqrt{c^2-b^2};\ b=\sqrt{c^2-a^2};\ c=\sqrt{a^2+b^2}.$$

If the proposed numbers express the numerical values of two sides, the number given by the appropriate formula expresses the numerical value of the third side.

c. In right-angled triangles, if the magnitudes of the acute angles are constant, the ratios of the sides are constant.



Let A B C be a triangle having the angle C, a right angle: through C′, any point in A C or A C produced, let C′B′ be drawn parallel to C B, and let the intersection of C′B′ and A B, or A B produced, be the point B′.

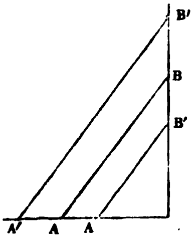
The triangles A B C, A B′C′, are equiangular (Euc. I. 29.): therefore,

$$\text{AB : BC :: AB}' : \text{B}'\text{C}' \text{ (Euc. VI. 4.)} \therefore \frac{c}{a}=\frac{c'}{a'};\ \frac{a}{c}=\frac{a'}{c'}$$

$$\text{AB : AC :: AB}' : \text{AC}' \quad \ldots\ldots \quad \frac{c}{b}=\frac{c'}{b'};\ \frac{b}{c}=\frac{b'}{c'}$$

$$\text{AC : BC :: AC}' : \text{B}'\text{C}' \quad \ldots\ldots \quad \frac{b}{a}=\frac{b'}{a'};\ \frac{a}{b}=\frac{a'}{b'}.$$

If, instead of drawing C′B′ parallel to C B, A′B′ is drawn parallel to A B, the triangles A B C, A′B′C, are equiangular, and, as before,



$$\text{AB : BC :: A}'\text{B}' : \text{B}'\text{C} \therefore \frac{c}{a}=\frac{c'}{a'};\ \frac{a}{c}=\frac{a'}{c'};$$

$$\text{AB : AC :: A}'\text{B}' : \text{A}'\text{C} \therefore \frac{c}{b}=\frac{c'}{b'};\ \frac{b}{c}=\frac{b'}{c'};$$

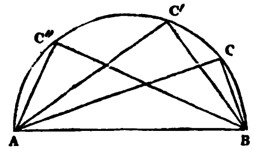
$$\text{AC : BC :: A}'\text{C : B}'\text{C} \therefore \frac{b}{a}=\frac{b'}{a'};\ \frac{a}{b}=\frac{a'}{b'}.$$

Consequently, although the absolute values of the sides of the right-angled triangles A B C, A B′C′; A B C, A′B′C, are changed, the ratios of the sides remain constant, while the acute angles are constant.

d. In right-angled triangles, if the magnitudes of the acute angles are changed, the ratios of the sides are made to vary.

In order that the changes in the values of the ratios of the sides, dependent on the variation of the magnitudes of the acute angles, may be distinctly traced, it will be convenient to make one of the sides of the triangle A B C constant.

If A B, the side opposite the right angle, is made constant; on A B describe a semicircle; and let a straight line A C, be placed on A B, and made to turn round the point A as a centre. Also, let A C, A C′, A C″, be the revolving line in different positions, and let B C, B C′, B C″, be joined.



Then as A C is turned round in the direction B C C′ C″ A, the angle A is increased from 0° to 90°;

the side A C or b is diminished from A B $= c$, to 0,
the side B C or a is increased from 0 to B A $= c$.

Consequently (since the side A B or c is constant), if the angle A is increased from 0 to 90°,

the ratio $\frac{\mathrm{CB}}{\mathrm{AB}}$ or $\frac{a}{c}$ is increased from $\frac{0}{c}$ to $\frac{c}{c}$ or from 0 to 1;

the ratio $\frac{\mathrm{AC}}{\mathrm{AB}}$ or $\frac{b}{c}$ is diminished from $\frac{c}{c}$ to $\frac{0}{c}$ or from 1 to 0;

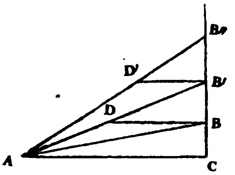
the ratio $\frac{\mathrm{BC}}{\mathrm{AC}}$ or $\frac{a}{b}$ is increased from $\frac{0}{c}$ to $\frac{c}{0}$ or from 0 to ∞;

the ratio $\frac{\mathrm{AB}}{\mathrm{CB}}$ or $\frac{c}{a}$ is diminished from $\frac{c}{0}$ to $\frac{c}{c}$ or from ∞ to 1;

the ratio $\frac{\mathrm{AB}}{\mathrm{AC}}$ or $\frac{c}{b}$ is increased from $\frac{c}{c}$ to $\frac{c}{0}$ or from 1 to ∞;

And the ratio $\frac{\mathrm{AC}}{\mathrm{BC}}$ or $\frac{b}{a}$ is diminished from $\frac{c}{0}$ to $\frac{0}{c}$ or from ∞ to 0.

If A C, one of the sides about the right angle, is made constant, and A B, B C, are made to vary with the variation of the angle A; then C B′ is greater than C B; and therefore A B′ or $\sqrt{\mathrm{AC}^2 + \mathrm{CB}'^2}$ is greater than A B or $\sqrt{\mathrm{AC}^2 + \mathrm{CB}^2}$. Consequently, if the angle A is increased, A B or c is increased, and B C or a is increased;



whence (A C or b being constant) the ratio $\frac{b}{c}$ is diminished,

the ratio $\frac{b}{a}$ is diminished,

the ratio $\frac{c}{b}$ is increased,

the ratio $\frac{a}{b}$ is increased.

With respect to the ratios $\frac{a}{c}$, $\frac{c}{a}$, since a, c, are both increased or both diminished, according as the angle A is increased or diminished, in order to prove that the ratios $\frac{a}{c}$, $\frac{c}{a}$, are not constant, it becomes necessary to demonstrate that a, c, are not increased or diminished in the same proportion.

For this purpose let B D be drawn parallel to A C; then, since B C A is a right angle, each of the angles B A C, B′A C, is less than a right angle (Euc. I. 17.). But the angle B A C is equal to the angle A B D, and the angle B′A C to the angle B′D B (Euc. I. 29.), therefore each of the angles A B D, B′D B, is less than a right angle. Now the angles A D B, B′D B, are together equal to two right angles (Euc. I. 13.), therefore the angle A D B is greater than a right angle, and consequently greater than the angle A B D: whence A B is greater than A D (Euc. I. 19.).

Again, since A B is greater than A D, $\frac{AB}{BC}$ or $\frac{c}{a}$ is greater than $\frac{AD}{BC}$.

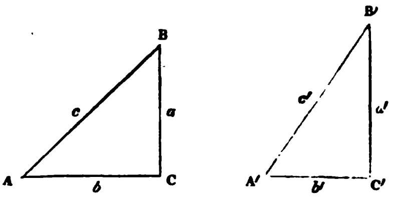
$$\text{But } \frac{AD}{BC} = \frac{AB'}{B'C} \text{ (Euc. VI. 2.).}$$

$$\therefore \frac{AB}{BC} > \frac{AB'}{B'C}, \text{ or } \frac{c}{a} > \frac{c'}{a'}.$$

In like manner it can be proved that $\frac{AB'}{B'C} > \frac{AB''}{B''C}$.

Lastly, since the ratio $\frac{c}{a} > \frac{c'}{a'}$, $\frac{a}{c}$, the reciprocal of $\frac{c}{a}$ is less than $\frac{a'}{c'}$, the reciprocal of $\frac{c'}{a'}$. Consequently, as the angle A is increased, the ratio $\frac{c}{a}$ is diminished, and the ratio $\frac{a}{c}$ increased.

e. If the ratio of any two sides of a right-angled triangle is made to vary, the magnitudes of the acute angles must vary. For, let it be supposed that the ratio $\frac{a}{c}$ differs from the ratio $\frac{a'}{c'}$, but that the angle A is equal to the angle A′, and by consequence B equal to B′.



Then, since the triangles A B C, A′B′C′, are equiangular $\frac{a}{c} = \frac{a'}{c'}$: but this is contrary to the hypothesis; therefore the angle A is not equal to the angle A′, nor B to B′.

f. From the preceding part of this article it follows that if in any right-angled triangle the ratio of two sides is given, all the ratios which can be formed with the sides, taken two and two, can be determined from the given ratio: and that a mutual dependence subsists between the acute angles and these ratios, such that if either is made to vary, a variation is necessarily produced in the other.

Consequently, to every particular value of either of the acute angles of a right-angled triangle, from 0 to 90°, correspond particular values of the ratios of the sides, and conversely. Whence, if the ratios of the sides are computed for all angles from 0 to 90° and arranged in tables, in connection with the angles, it will be possible, with the help of these tables, to find the ratio of any two sides of a right-angled triangle when an acute angle is given, and to find the values of the acute angles when the ratio of any two sides is given.

The calculation of the values of the ratios of the sides of a right-angled triangle corresponding to the different values of an acute angle from 0 to 90° is made by methods which are explained in Section IV.

In any calculation in which it is requisite to combine the numerical values of the sides and angles of triangles, the values of these ratios are substituted for the numerical values of the corresponding angles. By this expedient, the numbers which represent the sides and angles are referred to the same species of unit. (See the calculations in Sect. V.)

17. The ratios of the sides of the right-angled triangle A B C are distinguished by the following names: —

The ratio $\frac{a}{c}$ is termed the sine of the angle A

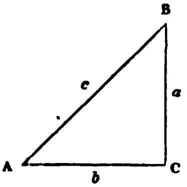
. . $\frac{a}{b}$ . . . tangent . .

. . $\frac{c}{b}$ . . . secant . .

. . $\frac{b}{c}$ . . . sine of the angle B

. . $\frac{b}{a}$ . . . tangent . .

. . $\frac{c}{a}$ . . . secant . .



The angle B being the complement of the angle A (Art. 15.), the ratios $\frac{b}{c}$, $\frac{b}{a}$, $\frac{c}{a}$, are also respectively the sine, the tangent, the secant, of the complement of the angle A; expressions which are abridged into cosine, cotangent, cosecant of the angle A.

In like manner the ratios $\frac{a}{c}$, $\frac{a}{b}$, $\frac{c}{b}$, are respectively the cosine, the cotangent, and the cosecant of the angle B.

The expressions, sine of the angle A, tangent of the angle A, &c., are written sin. A, tan. A, sec. A, cos. A, cot. A, cosec. A.

$1 - \cos. A$, or $1 - \frac{b}{c}$, is termed the versed sine of the angle A.

$1 - \sin. A$, or $1 - \frac{a}{c}$, is termed the versed sine of the angle B, or the versed sine of the complement of the angle A.

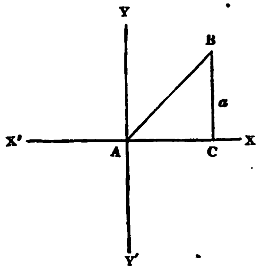
$1 + \cos. A$, or $1 + \frac{b}{c}$, is termed the versed sine of the supplement of the angle A.

Versed sine of the angle A is written versin. A; versed sine of the complement of the angle A, co. versin. A; versed sine of the supplement of the angle A, su. versin. A.

Powers of the expressions sin. A, cos. A, &c. are written $\sin.^2 A$; $\cos.^2 A$; $\sin.^3 A$; $\sin.^n A$; or, more correctly $(\sin. A)^2$; $(\cos. A)^2$; $(\sin. A)^3$; $(\sin. A)^n$, &c.

$\frac{a}{c}$, $\frac{a}{b}$, $\frac{c}{b}$, &c., are termed goniometrical or trigonometrical ratios.

18. The angle A being made to increase from 0 to 360°, it is required to find the consequent changes in the sign and value of sin. A.



Let the rectangular axes AX, AY, be drawn, and let the variable angle A be contained by the axis AX and the line AB: AB being of a constant length, and made to turn round the point A as a centre.

a. Beginning in the angle XAY; if AB is placed on AX and turned round until it coincides with AY, the angle A is increased from 0 to 90°. Let AB (in the annexed diagram) be the revolving line in any position intermediate between AX, AY, and let BC be drawn perpendicular to AX.

1st. To find the sign of the ratio $\frac{a}{c}$. The sign of the revolving line AB, which does not coincide in direction with either axis, is always to be considered $+$: whence, assuming that in the angle XAY the sign of BC is $+$, the sign of the ratio $\frac{\text{BC}}{\text{AB}}$ or $\frac{a}{c}$ is $+$.

2d. To find the change in the value of sin. A.

If AB coincides with AX, the angle $\text{A} = 0$, and BC or $a = 0$, $\therefore$ $\sin. 0 = \frac{o}{c} = 0$.

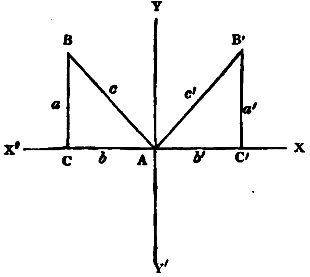
If AB is turned round from AX in the direction AY, the angle A is increased, and the perpendicular BC or a is increased: therefore, since c is constant, $\sin. \text{A} = \frac{a}{c}$ is increased.

If AB falls upon AY, BC coincides with AB, and is equal to it. In this position $\text{A} = 90°$, and $a = c$, $\therefore$ $\sin. 90° = \frac{a}{c} = \frac{c}{c} = 1$.

Whence the angle A being increased from 0 to 90°, sin. A is increased from 0 to $+1$.

b. Let AB revolve through the angle YAX′, and the angle A increase from 90° to 180°. Draw BC perpendicular to AX′.

Then, 1st. The sign of $\frac{a}{c}$ is $+$, because the sign of BC or a is $+$. (Art. 3.).



2d. If the revolving line AB is turned from AY in the direction AX′, the angle XAB or A is increased, and the perpendicular BC or a is diminished: therefore, also, $\sin. \text{A} = \frac{a}{c}$, is diminished.

If AB coincides with AX′, A $= 180°$, and BC or $a = 0$.

Therefore, sin. $180° = \frac{o}{c} = 0$.

Whence the angle A, being increased from 90° to 180°, sin. A is diminished from $+1$ to 0.

Let the angle B′AC′ be equal to the angle BAC: then XAB + B′AC′ $= 180°$ (Euc. I. 13.); and XAB $= 180° -$ B′AC′, or B′AC′ $= 180° -$ XAB.

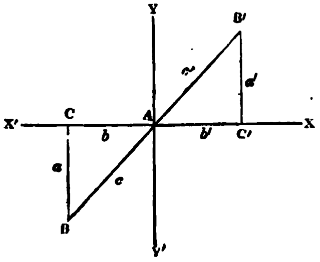
Again, if B′AC′ = BAC, and AB′ or $c' =$ AB or c, B′C = BC or $a' = a$.

$$\therefore \frac{a}{c} = \frac{a'}{c'}, \text{ or sin. XAB} = \text{sin. B'AC'}.$$

Consequently, the angles XAB, B′AC′, which are each the supplement of the other, have the same sine, which is + for both angles.

$$\therefore \text{sin. A} = \text{sin. } (180° - \text{A}).$$

c. Let AB fall in the angle X′AY′, and the angle A increase from 180° to 270°. Draw BC perpendicular to AX′.



1st. It having been assumed that the direction AY is positive, the direction AY′, is negative (Art. 3.): therefore the sign of BC or a is $-$; and the sign of the ratio $\frac{a}{c}$, also $-$.

2d. If the revolving line AB is moved from AX′ in the direction AY′, A is increased: the perpendicular BC or a is also increased in absolute value, and therefore, being negative, is diminished in relative value (Alg. Art. 3.). Therefore, also, sin. A $= \frac{-a}{c}$ is diminished.

If AB coincides with AY′, A $= 270°$, and BC = AB or $a = c$.

$$\therefore \text{sin. } 270° = \frac{-a}{c} = -1.$$

Consequently, if A is increased from 180° to 270°, sin. A is diminished from 0 to -1.

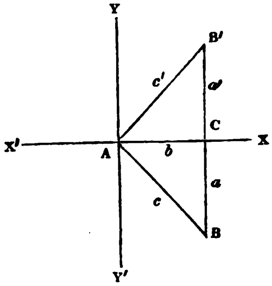
If $\angle$ BAC $= \angle$ B′AC′, and AB or $c =$ AB′ or c', $a = a' \therefore \frac{a}{c} = \frac{a'}{c'}$.

Whence the absolute quantities sin. (180° + A) and sin. A are equal.

The sign of sin. (180° + A) being negative, it follows that

$$\sin. (180° + A), = -\sin. A,$$
$$\text{or } \sin. A = -\sin. (180° + A).$$

d. Let AB fall in the angle XAY′, and the angle A increase from 270° to 360°. Draw BC perpendicular to AX.



1st. The sign of the ratio $\frac{a}{c}$ is −, because the sign of a is −.

2d. If the line AB is made to revolve from AY′ towards AX, the angle A is increased: the perpendicular a is diminished in absolute value, and, therefore, being negative, it is also increased in relative value.

Whence $\sin. A. = \frac{-a}{c}$ is increased in relative value.

If AB coincides with AX, $A = 360°$, and $a = 0$, ∴ $\sin. 360° = \frac{o}{c} = 0$.

Consequently A being increased from 270° to 360°, sin. A is increased from −1 to 0.

If the angle BAX is equal to the angle B′AX, $a = a'$, and $\frac{a}{c} = \frac{a'}{c'}$.

Therefore the absolute quantities sin. (360° − A) and sin. A are equal: the sign of the former being −,

$$\sin. (360° - A) = -\sin. A,$$
$$\text{or } \sin. A = -\sin. (360° - A).$$

e. If the angle CAB is traced by the line AB turning from AX in the direction AY′, it is negative (Art. 13. *a.*).

Whence if $CAB' = CAB$, $\frac{a'}{c'} = -\frac{a}{c}$.

$$\text{But } \frac{a'}{c'} = \sin. A, \text{ and } \frac{a}{c} = \sin. (-A).$$

$$\therefore \sin. A = -\sin. (-A).$$

f. From this discussion it appears that if the angle A is between

0 and 90°, sin. A is between	0 and 1;	its sign being +.
90° and 180°, . .	1 and 0	. . +.
180° and 270°, . .	0 and 1	. . −.
270° and 360°, . .	1 and 0	. . −.

19. If the angle A (fig. Art. 18. *a.*) is increased by 360°, the values of a, c, and therefore of $\frac{a}{c}$, are not altered by the change.

$$\therefore \text{sin. A} = \text{sin. } (360° + \text{A}).$$

For the same reasons,

$$\text{sin. A} = \text{sin. } (2 \times 360° + \text{A})$$
$$\text{sin. A} = \text{sin. } (n \times 360° + \text{A}); \text{ } n \text{ being any integer.}$$

Or, since $n \times 360° = 2n \times 180°$; $\text{sin. A} = \text{sin. } (2n \times 180° + \text{A})$.

Again, since $\text{sin. A} = \text{sin. } (180° - \text{A})$; (Art. 18. *b.*)

and $\text{sin. } (180°-\text{A}) = \text{sin. } (360° + (180° - \text{A})) = \text{sin. } (n\,.\,360° + (180° - \text{A})) = \text{sin. } (2n\,.\,180° + (180° - \text{A})) = \text{sin. } ((2n+1)\,180 - \text{A})$

$$\therefore \text{sin. A} = \text{sin. } \{(2n+1)180° - \text{A}\}.$$

Similarly, from Art. 18. *c*, *d*, it is found that

$$\text{sin. A} = -\text{sin. } \{(2n+1)180° + \text{A}\}.$$
$$\text{sin. A} = -\text{sin. } \{2n\,.\,180° - \text{A}\}.$$

Hence, if an even number ($2n$) of angles of 180° is added to any angle A, the sign and value of the sine are not changed: but, if an odd number ($2n+1$), the absolute value of the sine is the same, but its sign is changed.

Besides the values of sin. A found in this and the preceding articles, others may be determined.

a. A being any angle, A and $90° - \text{A}$ are complements of each other (Art. 15.),

$$\therefore \text{sin. A} = \text{cos. } (90° - \text{A}). \text{ (Art. 17.)}$$

b. $\text{Cosec. A} = \frac{c}{a}$. (Art. 17.).

$$\therefore 1 \div \text{cosec. A} = 1 \div \frac{c}{a} \text{ or } \frac{1}{\text{cosec. A}} = \frac{a}{c}$$

But $\text{sin. A} = \frac{a}{c}$, $\therefore \text{sin. A} = \frac{1}{\text{cosec. A}}$.

c. $\text{Sin. A} = \frac{a}{c}$ and $\text{cos. A} = \frac{b}{c}$ $\therefore \frac{\text{sin. A}}{\text{cos. A}} = \frac{a}{b}$.

But $\text{tan. A} = \frac{a}{b}$

$$\therefore \frac{\text{sin. A}}{\text{cos. A}} = \text{tan. A, and sin. A} = \text{cos. A tan. A}.$$

d. Since $\sin. A = \frac{a}{c}$, $\sin.{}^2 A = \frac{a^2}{c^2}$; and since $\cos. A = \frac{b}{c}$, $\cos.{}^2 A = \frac{b^2}{c^2}$.

$$\therefore \sin.{}^2 A + \cos.{}^2 A = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2}.$$

But $a^2 + b^2 = c^2$ (Euc. I. 47.).

$$\therefore \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

$$\text{And } \sin.{}^2 A + \cos.{}^2 A = 1.$$

$$\text{Whence } \sin.{}^2 A = 1 - \cos.{}^2 A.$$

$$\text{And } \sin. A = \sqrt{1 - \cos.{}^2 A}.$$

Collecting these different values of the sine of the angle A,

$$\begin{aligned}\text{Sin. } A &= -\sin.(-A) = \cos.(90^\circ - A) = \sin.(180^\circ - A) = -\sin.(180^\circ + A) = -\sin.(360^\circ - A) \\ &= \sin.(2n.180^\circ + A) = \sin.\{(2n+1)\,180^\circ - A\} = -\sin.\{(2n+1)\,180^\circ + A) \\ &= -\sin.(2n.180 - A) = \frac{1}{\text{cosec. } A} = \cos. A.\tan. A = \sqrt{1 - \cos.{}^2 A}.\end{aligned}$$

20. The angle A being made to increase from 0 to 360°, it is required to find the consequent changes in the sign and value of tan. A.

a. Angle A increasing from 0 to 90°. (Fig. Art. 18. *a.*)

1st. In the angle XAY, the signs of BC and AC or a and b are, by convention, positive; therefore the sign of the ratio $\frac{a}{b}$ is +.

2d. If AB coincides with AX, $A = 0$, and $a = o$,

$$\therefore \tan. 0 = \frac{o}{b} = 0.$$

If AB is turned round from AX in the direction AY, A is increased: a is also increased, and b diminished:

$$\therefore \tan. A = \frac{a}{b} \text{ is increased.}$$

When AB coincides with AY, $A = 90^\circ$; $a = c$; $b = o$,

$$\therefore \tan. 90^\circ = \frac{a}{o} = \infty.$$

Consequently, A being increased from 0 to 90°, tan. A is increased from 0 to $+\infty$.

If $A = 45^\circ$, $BC = AC$ or $a = b$, $\therefore \frac{a}{b} = 1$ $\therefore$ $\tan. 45^\circ = 1$.

b. The angle A increasing from 90° to 180°. (Fig. Art. 18. *b.*)

1st. In the angle YAX′, the sign of a is +, and the sign of b, −; therefore the sign of the ratio $\frac{a}{b}$ is −.

2d. By the rotation of AB from AY towards AX′, A and b are increased, and a is diminished. The absolute value of $\frac{a}{b}$ is consequently diminished; but the sign of the ratio $\frac{a}{b}$ being $-$, its relative value is increased.

When AB coincides with AX′, $A = 180$; $a = o$; $b = c$.

$$\therefore \text{ Tan. } 180° = \frac{o}{c} = 0.$$

Whence A being increased from 90° to 180°; tan. A is increased from $-\infty$ to 0.

It is to be observed that, tan. A passing through ∞, the sign of tan. A is changed from + to $-$.

If the angles BAC, B′AC′ make two right angles, $BC = B'C'$ and $AC = AC'$, or $a = a'$, $b = b'$. Therefore, $\frac{a}{b} = \frac{a'}{b'}$, or (regard being had to the difference of sign) $\frac{a}{b} = -\frac{a'}{b'}$. Whence, tan. $(180° - A) = -$ tan. A, or tan. $A = -$ tan. $(180° - A)$.

c. The angle A increasing from 180° to 270°. (Fig. Art. 18. *c.*)

1st. In the angle X′AY′, the sign of a is $-$, and the sign of b, $-$; therefore, the sign of the ratio $\frac{a}{b}$ is $+$.

2d. If AB is turned round from AX′ towards AY′, A and a are increased, but b is diminished. Whence, tan. $A = \frac{a}{b}$ is increased.

When AB coincides with AY′; $A = 270°$, $a = c$, $b = o$.

$$\therefore \text{ Tan. } 270° = \frac{a}{o} = \infty.$$

Consequently, A being increased from 180° to 270°, tan. A is increased from 0 to $+\infty$.

In this case, when tan. A passes through 0, the sign of tan. A is changed from $-$ to $+$.

If $BAC = B'AC'$, it is evident that $\frac{a}{b} = \frac{a'}{b'}$.

$$\text{or tan. } (180° + A) = \text{tan. } A.$$

d. The angle A increasing from 270° to 360°. (Fig. Art. 18. *d.*)

1st. In the angle XAY′, the sign of a is $-$, and the sign of b, $+$; therefore, the sign of the ratio $\frac{a}{b}$ is $-$.

2d. AB being turned round from AY′ to AX, A and b are increased, but a is diminished. The absolute value of $\frac{a}{b}$ is consequently diminished, and hence (its sign being $-$) its relative value is increased.

When AB coincides with AX, $\text{A} = 360°$; $a = o$; $b = c$.

$$\therefore \text{ Tan. } 360° = \frac{o}{c} = 0.$$

Consequently, A being increased from 270° to 360°, tan. A is increased from $-\infty$ to 0.

Again, when tan. A passes through ∞, the sign of tan. A is changed from $+$ to $-$.

If $\text{BAC} = \text{B}'\text{AC}$, $a = a'$, $b = b$, $\therefore \frac{a}{b} = -\frac{a'}{b}$, or $\tan.(360° - \text{A}) = -\tan.\text{A}$.

Hence also $\tan.(-\text{A}) = -\tan.\text{A}$, or $\tan.\text{A} = -\tan.(-\text{A})$.

Consequently when A is between 0 and 90°, tan. A is between 0 and ∞, its sign being $+$;
when A is between 90° and 180°, tan. A is between ∞ and 0, its sign being $-$;
when A is between 180° and 270°, tan. A is between 0 and ∞, its sign being $+$;
when A is between 270° and 360°, tan. A is between ∞ and 0, its sign being $-$.

21. If the angle A (Fig. Art. 18. *a.*) is increased by 360°; a, b, and therefore $\frac{a}{b}$ or tan. A are not altered by the change.

$$\therefore \text{ Tan. A} = \tan.(360° + \text{A}).$$

For the same reason, $\tan.\text{A} = \tan.(2 \times 360° + \text{A}) = \tan.(3 \times 360° + \text{A}) = \tan.(n.360° + \text{A})$

or, since $n.360 = 2n.180$; $\tan.\text{A} = \tan.(2n.180 + \text{A})$.

In the same manner it is proved that,

$$\text{Tan. A} = -\tan.\{(2n+1)\,180° - \text{A}\}$$
$$\text{Tan. A} = \tan.\{(2n+1)\,180° + \text{A}\}$$
$$\text{Tan. A} = -\tan.(2n.\,180° - \text{A}\}$$

a. If the algebraic sum of the angles A, B, is 90°;

$$\tan.\text{A} = \cot.\text{B}. \quad (\text{Art. } 17.)$$

$$\text{But B} = 90° - \text{A}, \quad \therefore \text{ Tan. A} = \cot.(90° - \text{A}).$$

b. It is proved in Art. 19. *c.* that $\tan.\text{A} = \frac{\sin.\text{A}}{\cos.\text{A}}$: and in Art. 19. *d.* that $\sin.^2\text{A} + \cos.^2\text{A} = 1$. Hence, $\sin.\text{A} = \sqrt{1 - \cos.^2\text{A}}$, and $\cos.\text{A} = \sqrt{1 - \sin.^2\text{A}}$; and consequently,

$$\tan.\text{A} = \frac{\sin.\text{A}}{\sqrt{1 - \sin.^2\text{A}}} = \frac{\sqrt{1 - \cos.^2\text{A}}}{\cos.\text{A}}$$

c. Since sec. $\mathrm{A} = \frac{c}{b}$, $\sec.^2 \mathrm{A} = \frac{c^2}{b^2}$; and since tan. $\mathrm{A} = \frac{a}{b}$, $\tan.^2 \mathrm{A} = \frac{a^2}{b^2}$.

$$\therefore \ \mathrm{Sec.}^2 \mathrm{A} - \tan.^2 \mathrm{A} = \frac{c^2}{b^2} - \frac{a^2}{b^2} = \frac{c^2 - a^2}{b^2}.$$

$$\text{But } c^2 - a^2 = b^2. \quad \text{(Euc. i. 47.)}$$

$$\therefore \ \mathrm{Sec.}^2 \mathrm{A} - \tan.^2 \mathrm{A} = \frac{b^2}{b^2} = 1.$$

$$\mathrm{Tan.}^2 \mathrm{A} = \sec.^2 \mathrm{A} - 1$$

$$\text{And } \tan. \mathrm{A} = \sqrt{\sec.^2 \mathrm{A} - 1}.$$

d. Cot. $\mathrm{A} = \frac{b}{a}$. (Art. 17.) $\therefore \frac{1}{\cot. \mathrm{A}} = \frac{a}{b}$; but $\tan. \mathrm{A} = \frac{a}{b} \therefore \tan. \mathrm{A} = \frac{1}{\cot. \mathrm{A}}$.

e. The different values of the tangent of the angle A, found in Articles 20 and 21 are

$$\begin{aligned} \mathrm{Tan\ A} &= -\tan. (-\mathrm{A}) = \cot. (90^\circ - \mathrm{A}) = -\tan. (180^\circ - \mathrm{A}) = \tan. (180^\circ + \mathrm{A}) \\ &= -\tan. (360^\circ - \mathrm{A}) = \tan. (2n . 180 + \mathrm{A}) = -\tan. \left\{(2n+1)\, 180^\circ - \mathrm{A}\right\} \\ &= \tan. \left\{(2n+1)\, 180^\circ + \mathrm{A}\right\} = -\tan. (2n . 180^\circ - \mathrm{A}) = \frac{\sin. \mathrm{A}}{\cos. \mathrm{A}} \\ &= \frac{\sin. \mathrm{A}}{\sqrt{1 - \sin.^2 \mathrm{A}}} = \frac{\sqrt{1 - \cos.^2 \mathrm{A}}}{\cos. \mathrm{A}} = \sqrt{\sec.^2 \mathrm{A} - 1} = \frac{1}{\cot. \mathrm{A}}. \end{aligned}$$

22. The angle A being increased from 0 to 360°, it is required to find the consequent changes in the sign and value of sec. A.

a. The angle A increasing from 0 to 90°. (Fig. Art. 18. *a.*)

1st. In the angle XAY, the sign of b is +; whence (the sign of c being always +) the sign of the ratio $\frac{c}{b}$ is +.

2d. When AB coincides AX, $\mathrm{A} = 0$, and $\mathrm{AC} = \mathrm{AB}$ or $b = c$.

$$\therefore \ \mathrm{Sec.}\ 0 = \frac{c}{b} = \frac{c}{c} = 1.$$

If AB is turned round from AX in the direction AY, A is increased, c is constant, and b is diminished; therefore, sec. $\mathrm{A} = \frac{c}{b}$ is increased.

When AB coincides with AY, $\mathrm{A} = 90^\circ$, $b = 0$, and

$$\mathrm{Sec.}\ 90^\circ = \frac{c}{o} = \infty.$$

Whence, A being increased from 0 to 90°, sec. A is increased from + 1 to + ∞.

b. The angle A increasing from 90° to 180°. (Fig. Art. 18. *b.*)

1st. In the angle YAX′, the sign of b is −, therefore the sign of $\frac{c}{b}$ is −.

2d. AB being turned round from AY in the direction

AX′, A is increased, and b is increased. Therefore the absolute value of sec. A $=\frac{b}{c}$, is diminished; but the sign of $\frac{c}{b}$ being —, the relative value of the expression is increased.

When AB coincides with AX′, A $= 180°$, and $b = c$.

$$\therefore \text{ Sec. } 180° = \frac{c}{-b} = -\frac{c}{c} = -1.$$

Whence, A increasing from 90° to 180°; sec. A increases from $-\infty$ to -1; the sign of sec. A having been changed from + to − in passing through ∞.

$$\text{If B}'\text{AC}' = \text{BAC}, \ \frac{c'}{b'} = -\frac{c}{b}, \text{ or sec. A} = -\text{sec } (180° - \text{A}).$$

c. The angle A increasing from 180° to 270°. (Fig. Art. 18. *c.*)

1st. In the angle X′AY′, the sign of b is —, therefore the sign of $\frac{c}{b}$ is —.

2d. AB being turned from AX′ in the direction AY′, A is increased, and b diminished; therefore sec. A $=\frac{c}{b}$ is increased in absolute, but (the sign being —) diminished in relative value.

When AB coincides with AY′, A $= 270°$, and $b = 0$;

$$\therefore \text{ Sec. } 270° = -\frac{c}{o} = -\infty.$$

Consequently, A being increased from 180° to 270°, sec. A is diminished from -1 to $-\infty$.

$$\text{If B}'\text{AC}' = \text{BAC}, \ \frac{c'}{b'} = -\frac{c}{b}, \text{ or sec. A} = -\text{ sec. } (180° + \text{A}).$$

d. The angle A increasing from 270° to 360° (Fig. Art. 18. *d.*)

1st. In the angle XAY′, the sign of b is +, therefore the sign of $\frac{c}{b}$ is +.

2d. AB being turned round from AY′ in the direction AX, A is increased and b is increased: therefore sec. A $=\frac{c}{b}$ is diminished.

When AB coincides with AX, A $= 360°$ and $b = c$.

$$\therefore \text{ Sec. } 360° = \frac{c}{b} = \frac{c}{c} = 1.$$

Whence, if A is increased from 270° to 360°, sec. A is diminished from $+\infty$ to $+1$; the sign of sec. A being changed in passing through ∞.

$$\text{If } \mathrm{B'AC} = \mathrm{BAC},\ \frac{c'}{b} = \frac{c}{b},\ \text{or sec. A} = \text{sec. } (360^\circ - \text{A}).$$

Also, if the angle CAB is considered negative, sec. A = sec. (— A).

Consequently when A is between 0 and 90°, sec. A is between 1 and ∞, its sign being + ;
when A is between 90° and 180°, sec. A is between ∞ and 1, its sign being —;
when A is between 180° and 270°, sec. A is between 1 and ∞, its sign being —;
when A is between 270° and 360°, sec. A is between ∞ and 1, its sign being +.

23. It can be proved, as for the sine and tangent in Articles 19 and 21, that

$$\begin{aligned} \text{Sec. A} &= \text{sec. } (2n.180^\circ + \text{A}) \\ &= -\text{sec. } \{(2n+1)\,180^\circ - \text{A}\} \\ &= -\text{sec. } \{(2n+1)\,180^\circ + \text{A}\} \\ &= \quad \text{sec. } (2n.180^\circ - \text{A}). \end{aligned}$$

a. If the angles A and B are complements of each other,

$$\text{Cosec. B} = \text{sec. A}: \text{ but B} = 90^\circ - \text{A}$$

$$\therefore\ \text{Sec. A} = \text{cosec. } (90^\circ - \text{A}).$$

b. Since $\frac{b}{c} = \text{cos. A},\ \frac{c}{b} = \frac{1}{\text{cos. A}}$. But $\frac{c}{b} = \text{sec. A}$

$$\therefore\ \text{Sec. A} = \frac{1}{\text{cos A}}.$$

c. Since (Art. 19. *d.*) $\text{cos.}^2\text{A} = 1 - \text{sin.}^2\text{A}, \text{cos. A} = \sqrt{1 - \text{sin.}^2\text{A}}$

$$\therefore\ \frac{1}{\text{cos. A}} = \frac{1}{\sqrt{1 - \text{sin.}^2\text{A}}},\ \text{or sec. A} = \frac{1}{\sqrt{1 - \text{sin.}^2\text{A}}}.$$

d. Since (Art. 21. *b*), $\text{sec.}^2\text{A} - \text{tan.}^2\text{A} = 1$, $\text{sec.}^2\text{A} = 1 + \text{tan.}^2\text{A}$,

$$\text{And, sec. A} = \sqrt{1 + \text{tan.}^2\text{A}}.$$

e. The different values of the secant of the angle A, found in Articles 22 and 23 are,

$$\begin{aligned} \text{Sec. A} &= \text{sec. } (-\text{A}) = \text{cosec. } (90^\circ - \text{A}) = -\text{sec. } (180^\circ - \text{A}) = -\text{sec. } (180^\circ + \text{A}) = \text{sec. } (360^\circ + \text{A}) \\ &= \text{sec. } (2n.180^\circ + \text{A}) = -\text{sec.}\{(2n+1)180^\circ - \text{A}\} = -\text{sec.}\{(2n+1)180^\circ + \text{A}\} = \text{sec. } (2n.180^\circ - \text{A}) \\ &= \frac{1}{\text{cos. A}} = \frac{1}{\sqrt{1 - \text{sin}^2\text{A}}} = \sqrt{1 + \text{tan.}^2\text{A}}. \end{aligned}$$

24. The changes in the sign and value of cos. A consequent upon changes in the value of the angle A, may be determined (as in the cases of sin. A, tan. A, sec. A), from the figures of Art. 18.; or from the values of the expression $\frac{1}{\text{sec. A}}$, which is

equal to cos. A (Art. 23. *b.*). From the expression $\frac{1}{\text{sec. A}}$ are found the values of cos. A, exhibited in the following table.

A being increased continuously from	sec. A is made to vary between	Therefore cos. A is made to vary between
0 to 90°	$+1$ and $+\infty$	$\frac{1}{+1}$ and $\frac{1}{+\infty}$ or $+1$ and 0.
90° to 180°	$-\infty$ and -1	$\frac{1}{-\infty}$ and $\frac{1}{-1}$ or 0 and -1.
180° to 270°	-1 and $-\infty$	$\frac{1}{-1}$ and $\frac{1}{-\infty}$ or -1 and 0.
270° to 360°	$+\infty$ and $+1$	$\frac{1}{+\infty}$ and $\frac{1}{+1}$ or 0 and $+1$.

a. It is evident, from observation of the figures of Article 18., that cos. A, cos. $(180° - \text{A})$, cos. $(180° + \text{A})$, cos. $(360° - \text{A})$, cos. $(-\text{A})$, are equal in absolute value; but that cos. $(180° - \text{A})$ and cos. $(180° + \text{A})$ are affected with the sign $-$.

b. If any number of revolutions ($n.360°$ or $2\,n.180°$) are added to each of these expressions of the cosine of the angle A, it is found that cos. A $=$ cos. $(2\,n.180° + \text{A}) = -$ cos. $\{(2\,n + 1)\,180° - \text{A}\} = -$ cos. $\{(2\,n + 1)\,180° + \text{A}\} =$ cos. $(2\,n.180 - \text{A})$.

Other values of cos. A are,

c. Cos. A $=$ sin. $(90° - \text{A})$. (Art. 17.)

d. Cos. A $= \frac{1}{\text{sec. A}}$. (Art. 24.)

e. Since cos. A $= \frac{1}{\text{sec. A}}$ and sec. A $= \sqrt{1 + \tan.^2 \text{A}}$. (Art. 23. *d.*)

$$\therefore \cos. \text{A} = \frac{1}{\sqrt{1 + \tan.^2 \text{A}}}.$$

f. Cos. A $= \frac{\text{sin. A}}{\text{tan. A}}$, (Art. 19. *c.*) $= \frac{\text{sin. A}}{\sqrt{\sec.^2 \text{A} - 1}}$. (Art. 21. *c.*)

g. Cos. A $= \sqrt{1 - \sin.^2 \text{A}}$. (Art. 19. *d.*)

25. The changes in the sign and value of cot. A can be, in like manner, determined from the expression $\frac{1}{\text{tan. A}}$, which is equal to tan. A (Art. 21. *d.*), thus,

A being increased continuously from	tan. A is made to vary between	Therefore cot. A is made to vary between
0 to 90°	0 and + ∞	$\frac{1}{0}$ and $\frac{1}{+\infty}$ or + ∞ and 0.
90° to 180°	− ∞ and 0	$\frac{1}{-\infty}$ and $\frac{1}{0}$ or 0 and − ∞.
180° to 270°	0 and + ∞	$\frac{1}{0}$ and $\frac{1}{+\infty}$ or + ∞ and 0.
270° to 360°	− ∞ and 0	$\frac{1}{-\infty}$ and $\frac{1}{0}$ or 0 and − ∞.

a. It may be shown as for the tangent of the angle A, in Art. 21., that $\cot. A = -\cot. (180° - A) = \cot. (180° + A) = -\cot. (360° - A) = -\cot. (-A) = \cot. (2\,n\,.\,180° + A) = -\cot. \{(2n + 1)\,180° - A\} = \cot. \{(2n + 1)\,180° + A\} = -\cot. (2\,n\,.\,180° - A)$.

b. Since $\tan. A = \frac{\sin. A}{\cos. A}$ (Art. 19. *c.*); $\frac{1}{\tan. A}$ or $\cot. A = \frac{\cos. A}{\sin. A}$.

c. Since A and $90° - A$ are complements of each other, $\cot. A = \tan. (90° - A)$.

d. Since $\text{cosec.}^2 A = \frac{c^2}{a^2}$ and $\cot.\,^2 A = \frac{b^2}{a^2}$, $\text{cosec.}^2 A - \cot.^2 A$

$$= \frac{c^2 - b^2}{a^2} = \frac{a^2}{a^2} = 1$$

$$\therefore \cot.^2 A = \text{cosec.}^2 A - 1, \text{ and } \cot. A = \sqrt{\text{cosec.}^2 A - 1}.$$

26. Lastly, the changes in the sign and value of cosec. A can be determined from the expression $\frac{1}{\sin. A}$ or $\frac{c}{a}$, which (Art. 19. *b.*) is equal to cosec. A.

A being increased continuously from	sin. A is made to vary between	Therefore cosec. A is made to vary between
0 to 90°	0 and + 1	$\frac{1}{0}$ and $\frac{1}{+1}$ or + ∞ and + 1.
90° to 180°	+ 1 and 0	$\frac{1}{+1}$ and $\frac{1}{0}$ or + 1 and + ∞.
180° to 270°	0 and − 1	$\frac{1}{0}$ and $\frac{1}{-1}$ or − ∞ and − 1.
270° to 360°	− 1 and 0	$\frac{1}{-1}$ and $\frac{1}{0}$ or − 1 and − ∞.

a. Denoting any number of complete revolutions by $n \times 360°$ or $2n \times 180°$, it is found, as for the sine of the angle A, in Article 19., that

$$\text{cosec. A} = \text{cosec.}\,(2n \,.\, 180° + \text{A}) = \text{cosec.}\left\{(2n+1)\,180° - \text{A}\right\} = -\,\text{cosec.}\left\{(2n+1)\,180° + \text{A}\right\} = -\,\text{cosec.}\,(2n \,.\, 180° - \text{A});$$

and, by making $n = 0$, in the last of these expressions, that cosec. $\text{A} = -\,\text{cosec.}\,(-\,\text{A})$.

b. Since A and $90° - \text{A}$ are complements of each other, cosec. $\text{A} = \text{sec.}\,(90° - \text{A})$.

c. Since cosec. $\text{A} = \dfrac{1}{\text{sin. A}}$, and sin. $\text{A} = \sqrt{1 - \text{cos.}^2\,\text{A}}$. (Art. 19. *d.*)

$$\therefore \text{ Cosec. A} = \frac{1}{\sqrt{1 - \text{cos.}^2\,\text{A}}}.$$

d. Since $\text{cosec.}^2\,\text{A} - \text{cot.}^2\,\text{A} = 1$ (Art. 25. *d.*), $\text{cosec.}^2\,\text{A} = 1 + \text{cot.}^2\,\text{A}$, and cosec. $\text{A} = \sqrt{1 + \text{cot.}^2\,\text{A}}$.

27. Since sin. $\text{A} = \dfrac{1}{\text{cosec. A}}$, sin. A cosec. $\text{A} = 1$;

$$\text{tan. A} = \frac{1}{\text{cot. A}},\ \text{tan. A cot. A} = 1;$$

$$\text{sec. A} = \frac{1}{\text{cos. A}},\ \text{sec. A cos. A} = 1.$$

28. By means of the preceding formulæ, all the six ratios can be expressed in terms of any one of the six. Let it, for example, be required to express tan. A, sec. A, cos. A, cot. A, cosec. A, in terms of sin. A.

$$\text{Tan. A} = \frac{\text{sin. A}}{\text{cos. A}} = \frac{\text{sin. A}}{\sqrt{1 - \text{sin.}^2\,\text{A}}}. \quad \text{(Art. 21. } b.)$$

$$\text{Sec. A} = \frac{1}{\sqrt{1 - \text{sin.}^2\,\text{A}}}. \quad \text{(Art. 23. } c.)$$

$$\text{Cos. A} = \sqrt{1 - \text{sin.}^2\,\text{A}}. \quad \text{Art. 24. } g.)$$

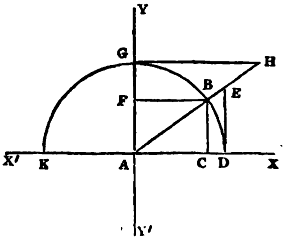
$$\text{Cot. A} = \frac{\text{cos. A}}{\text{sin. A}}\ \text{(Art. 25. } b.) = \frac{\sqrt{1 - \text{sin.}^2\,\text{A}}}{\text{sin. A}}\ \text{(because cos. A} = \sqrt{1 - \text{sin.}^2\,\text{A}}).$$

$$\text{Cosec. A} = \frac{1}{\text{sin. A}}. \quad \text{(Art. 19. } b.)$$

29. The limits of the variation of the trigonometrical quantities versin. A, coversin. A, suversin. A, are exhibited in the annexed table.

The angle A being increased from	0 to 90°	90° to 180°	180° to 270°	270° to 360°.
Sin. A varies between -	0 and +1	+1 and 0	0 and −1	−1 and 0.
Cos. A varies between -	+1 and 0	0 and −1	−1 and 0	0 and +1.
Therefore,				
versin A, which is equal to 1 − cos. A, varies between	0 and +1	+1 and +2	+2 and +1	+1 and 0.
coversin A, which is equal to 1 − sin. A, varies between	+1 and 0	0 and +1	+1 and +2	+2 and +1.
suversin A, which is equal to 1 + cos. A, varies between	+2 and +1	+1 and 0	0 and +1	+1 and +2.

30. The trigonometrical ratios are also defined as straight lines connected with any arc of a circle whose radius is 1. Let the axes AX, AY, and the straight line AB, which makes with AX the angle A, be drawn as in the figure of Art. 18. From A as centre with a radius equal to the unit of length, describe the circle DGK; from B, D, draw BC, DE perpendicular to AX: from B, G, draw BF, GH, perpendicular to AY; and produce AB, if necessary, to intersect the perpendicular from G in the point H. Then,



a. BC, the perpendicular let fall from one extremity of the arc BD, upon AD, the diameter which passes through the other extremity, is the sine of the arc BD.

The line BC falls wholly within the cavity of the arc BD (Euc. III. 2.): hence the name, sine, from *sinus*, hollow or concavity.

b. DE, the perpendicular drawn from the extremity of AD, the diameter which passes through one extremity of the arc BD and terminated at its intersection in E, with AB, the diameter which passes through the other extremity of the arc BD, is the tangent of the arc BD.

The line DE touches the arc in D (Euc. III. 16.): hence the name tangent or touching line.

c. AE, the straight line intercepted between the centre of the arc and that extremity of the tangent which is not in contact with the arc, is the secant of the arc BD.

The line AE cuts the circumference in B: hence the name secant or cutting line.

d. In like manner the lines BF, GH, AH, are named the sine, tangent, and secant respectively, of the arc BG. Or, if the arc BG (as in the diagram) is the complement of the arc BD; then BF, GH, AH, are termed respectively the cosine, cotangent, and cosecant of the arc BD.

e. Since BF = AC (Euc. I. 34.), AC may be taken as the cosine of the arc BD, and the cosine of an arc may be defined as the part of the radius intercepted between the centre of the arc and the foot of the sine of the arc.

f. CD, the part of the radius intercepted between the foot of the sine and the circumference is the versed sine of the arc BD.

Since AD = rad. = 1; AC = cos. of arc. BD; and CD = AD − AC; therefore versin. of arc. BD = 1 − cos. of arc BD.

g. FG, the part of the radius intercepted between the foot of the sine of the arc BG (which is the complement of the arc BD) and the extremity of the arc BG, is the versed sine of the arc BG, or the coversed sine of the arc BD.

Since AG = 1; AF = BC = sin. of arc BD; and FG = AG − AF; therefore coversin. of arc BD = 1 − sin. of arc BD.

h. KC, the part of the diameter intercepted between the extremity of the arc BGK (which is the supplement of BD) and the foot of the sine of the arc BD, is the su. versed sine of the arc BD.

KC = KA + AC : but KA = rad. = 1, and AC = cos. of arc BD: therefore su. versin. of arc BD = 1 + cos. of arc BD.

k. Under the hypothesis, radius = 1, the trigonometrical quantities sin., tan., &c. are expressed by the lines BC, DE, &c.; but the quantities sin. of arc BD, tan. of arc BD, &c., are still ratios, and their values are the numbers by which the lengths BC, DE, &c., are expressed in terms of the length AB or AD equal to 1.

l. If the radius AB is less or greater than 1, then

the sine of the arc BD is $\frac{BC}{AB}$

tan. . . . $\frac{DE}{AD}$

sec. . . . $\frac{AE}{AD}$

cos. . . $\frac{FB}{AB}$ or $\frac{AC}{AB}$

cot. . . . $\frac{GH}{AG}$,

cosec. . . . $\frac{AH}{AG}$.

Now the triangles ABC, AED, ABF, AHG, being equiangular and similar to each other (Euc. I. 29. and VI. 4.), and also to the figure of Art. 17., if the angle BAC is equal to the angle A of fig. Art. 17.; it follows that

$$
\begin{aligned}
&\frac{\mathrm{BC}}{\mathrm{AB}} = \frac{a}{c} = \sin.\ \mathrm{A} \\
&\frac{\mathrm{DE}}{\mathrm{AD}} = \frac{\mathrm{BC}}{\mathrm{AC}} = \frac{a}{b} = \tan.\ \mathrm{A} \\
&\frac{\mathrm{AE}}{\mathrm{AD}} = \frac{\mathrm{AB}}{\mathrm{AC}} = \frac{c}{b} = \sec.\ \mathrm{A} \\
&\frac{\mathrm{FB}}{\mathrm{AB}} = \frac{\mathrm{AC}}{\mathrm{AB}} = \frac{b}{c} = \cos.\ \mathrm{A} \\
&\frac{\mathrm{GH}}{\mathrm{AG}} = \frac{\mathrm{AC}}{\mathrm{BC}} = \frac{b}{a} = \cot.\ \mathrm{A} \\
&\frac{\mathrm{AH}}{\mathrm{AG}} = \frac{\mathrm{AB}}{\mathrm{BC}} = \frac{c}{a} = \mathrm{cosec.}\ \mathrm{A}.
\end{aligned}
$$

The sine, tangent, &c. of any angle and of that arc whose centre is at the vertex of the angle and whose extremities are limited by the sides of the angle, are consequently identical: in other words, an angle and an arc which contain the same number of degrees, minutes, &c., have the same sine, tangent, secant, &c.

l. Trigonometrical formulæ, in which the radius is equal to 1, are transformed into equivalent expressions in which the radius r is less or greater than 1 by dividing each sine, tangent, &c. of the formulæ related to radius $= 1$ by r; each $\sin.^2$, $\tan.^2$, &c. by r^2, and so on for higher powers of the sine, tangent, &c. (Art. 9. *b.*)

m. The circumference of a circle whose radius is 1, is denoted by $2\,\pi$ (Art. 8. *b.*): and the same symbol is frequently employed to express the number of degrees in its circumference. In this sense, $2\,\pi = 360°$; $\pi = 180°$; $\frac{\pi}{2} = 90°$; &c. Employing this notation, and denoting the arc which contains the same number of degrees as the angle A, by ϕ, certain formulæ of Articles 19. 21. 23, 24. *b*, 25. *a*, 26. *a*, become,

$$
\begin{aligned}
\sin.\varphi &= \cos.\left(\frac{\pi}{2}-\varphi\right) = \sin.(2n\pi+\varphi) = \sin.\{(2n+1)\pi-\varphi\} = -\sin.\{(2n+1)\pi+\varphi\} = -\sin.(2n\pi-\varphi). \\
\tan.\varphi &= \cot.\left(\frac{\pi}{2}-\varphi\right) = \tan.(2n\pi+\varphi) = -\tan.\{(2n+1)\pi-\varphi\} = \tan.\{(2n+1)\pi+\varphi\} = -\tan.(2n\pi-\varphi). \\
\sec.\varphi &= \mathrm{cosec.}\left(\frac{\pi}{2}-\varphi\right) = \sec.(2n\pi+\varphi) = -\sec.\{(2n+1)\pi-\varphi\} = -\sec.\{(2n+1)\pi+\varphi\} = \sec.(2n\pi-\varphi). \\
\cos.\varphi &= \sin.\left(\frac{\pi}{2}-\varphi\right) = \cos.(2n\pi+\varphi) = -\cos.\{(2n+1)\pi-\varphi\} = -\cos.\{(2n+1)\pi+\varphi\} = \cos.(2n\pi-\varphi). \\
\cot.\varphi &= \tan.\left(\frac{\pi}{2}-\varphi\right) = \cot.(2n\pi+\varphi) = -\cot.\{(2n+1)\pi-\varphi\} = \cot.\{(2n+1)\pi+\varphi\} = -\cot.(2n\pi-\varphi). \\
\mathrm{cosec.}\varphi &= \sec.\left(\frac{\pi}{2}-\varphi\right) = \mathrm{cosec.}(2n\pi+\varphi) = \mathrm{cosec.}\{(2n+1)\pi-\varphi\} = -\mathrm{cosec.}\{(2n+1)\pi+\varphi\} = -\mathrm{cosec.}(2n\pi-\varphi).
\end{aligned}
$$

n. It has been shown (Art. 8.) that if regular polygons of 12288 sides are described about and inscribed in a circle, their perimeters differ by nearly $\frac{2}{10000000}$ths of the radius. The length of a side of the circumscribed polygon therefore exceeds the length of a side of the inscribed polygon by $\frac{1}{12288}$ of $\frac{2}{10000000}$ths of the radius, and the difference in the lengths of the half sides of the polygons is equal to the half of this quantity.

If the number of sides of the two polygons is increased, the approximation of the ratio of their sides to a ratio of equality is also increased.

If a regular polygon whose side subtends an arc of 2″ be inscribed in and a similar polygon described about a circle, the number of sides, viz. 360 × 60 × 30 or 648000, being greater than in the instance of Article 8., the approximation of the sides of the polygons to equality with each other is also nearer.

The length of a half side of the circumscribed and inscribed polygons being considered equal, the circumference, or arc of 1″, which is less than the first and greater than the second (Circle, Principle,) must be also considered as equal to either. Consequently, if each of the three values be divided by that of the radius, the quotients must be equal: wherefore,

$$\frac{\text{Half side of circumscribed polygon of 648000 sides}}{\text{radius}}=\frac{\text{arc of 1}''}{\text{rad.}}=\frac{\text{half side of inscribed polygon of 648000 sides}}{\text{radius}}$$

or, $$\tan.\ 1''=\frac{\text{arc of } 1''}{\text{rad.}}=\sin.\ 1'',$$

or, making radius = 1

$$\tan.\ 1''=\text{arc of } 1''=\sin.\ 1'':$$

Hence, $$\frac{\tan.\ 1''}{\text{arc of } 1''}=1;\ \frac{\sin.\ 1''}{\text{arc of } 1''}=1:$$

And, generally, denoting an indefinitely small arc by h,

$$\frac{\tan.\ h}{h}=1;\ \frac{\sin.\ h}{h}=1.$$

o. Since tan. 1″ = arc of 1″; and the arc of 1″ is the $\frac{1}{1296000}$th part of the whole circumference;

$$1296000\times\tan.\ 1''=1296000\times\text{arc of }\frac{360^\circ}{1296000},$$

$$\text{or, } 1296000\times\tan.\ 1''=\text{circumference.}$$

Denoting, therefore, any arc by ϕ,
an indefinitely great number by n,
and, consequently, an indefinitely small arc by $\frac{\phi}{n}$,

$$n \tan. \frac{\phi}{n} = \phi.$$

p. The length of the arc of 1″, expressed in parts of the radius $= 1$, is given by the proportion

$$180^\circ : 1'' :: 3.1415926535 + : \frac{3.14159 \,\&\text{c.} \times 1}{180 \times 60 \times 60} = \frac{.0031415926535897932384626433 +}{648}$$
$$= .00000484813681107992783713371 + . \text{ (Art. 14. } e.)$$

This result may also be considered equal to the numerical value of either of the ratios tan. 1″, sin. 1″.

q. When small angles enter into trigonometrical computations, they are usually expressed in seconds of a degree, and frequently by the corresponding circular arcs, in terms of the radius considered as the unit of length; and it is convenient to express the equivalents of the angles in terms of the arcs, or the converse, by means of the computed length of an arc subtending an angle of one second, viz.

$$\frac{\pi}{180 \times 60 \times 60} \text{ or } \frac{1}{206264.8} \text{ or again } .0000048481368 +$$

Let this be expressed by arc 1″.

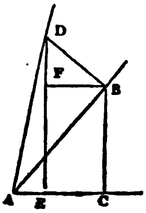
Then θ being an angle expressed in seconds, θ arc 1″ is the value of the corresponding circular arc, the radius being unity; and this is manifestly equivalent to $\frac{\theta}{206264.8}$.

Again, θ being the expression of a circular arc when the radius is unity, $\frac{\theta}{\text{arc } 1''}$ is the equivalent angle expressed in seconds.

For θ arc 1″, is frequently put θ sin. 1″; and for $\frac{\theta}{\text{arc } 1''}$, is put $\frac{\theta}{\sin. 1''}$.

31. Let the angle BAC contain A°, and the angle BAD, B°. From D, any point in AD, let DB be drawn perpendicular to AB, and DE perpendicular to AC: from B let BC be drawn perpendicular to AC, and BF perpendicular to DE; and produce DE, if necessary, to meet BF in F.

Fig. 1.



By this construction, FE = BC, and BF = CE (Euc. I. 34.); also, the angle BDF = FBA in fig. 1. (Euc. VI. 8.), and the supplement of FBA in fig. 2. (Euc. I. 13. and 32.) = BAC, (Euc. I. 29.) = A.

And by Article 17, $\frac{BC}{AB}$ or $\frac{BF}{BD}$ = sin. A

$\frac{AC}{AB}$ or $\frac{FD}{BD}$ = cos. A

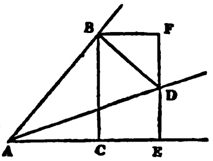
$\frac{BD}{AD}$ = sin. B

$\frac{AB}{AD}$ = cos. B

$\frac{DE}{AD}$ = sin. (A ± B)

$\frac{AE}{AD}$ = cos. (A ± B).

Fig. 2.



Of these quantities, let sin. A, cos. A, sin. B, cos. B, be given, and let it be required to express sin. (A ± B), cos. (A ± B) in terms of the sines and cosines of the angles A and B.

1st $\frac{DE}{AD}=\frac{EF \pm FD}{AD}=\frac{BC \pm FD}{AD}=\frac{BC}{AD}\pm\frac{FD}{AD}$.

But $\frac{BC}{AD}=\frac{BC}{AB}\times\frac{AB}{AD}$, (Alg. Art. 49.) = sin. A cos. B.

$\frac{FD}{AD}=\frac{FD}{BD}\times\frac{BD}{AD}$ = cos. A sin. B.

And $\frac{DE}{AD}$ = sin. (A ± B).

Whence sin. (A ± B) = sin. A cos. B ± cos. A sin. B.

2d. $\frac{AE}{AD}=\frac{AC \mp CE}{AD}=\frac{AC \mp BF}{AD}=\frac{AC}{AD}\mp\frac{BF}{AD}$:

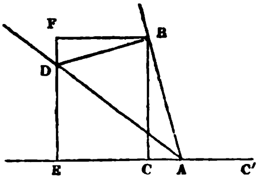
Now, $\frac{AC}{AD}=\frac{AC}{AB}\times\frac{AB}{AD}$ = cos. A cos. B.

And, $\frac{BF}{AD}=\frac{BF}{BD}\times\frac{BD}{AD}$ = sin. A sin. B.

Wherefore, since $\frac{AE}{AD}$ = cos. (A ± B),

Cos. (A ± B) = cos. A cos. B ∓ sin. A sin. B.

a. In the preceding investigation it has been assumed that each of the angles A, B, A+B, is less than a right angle.



Let now the angle C′AB be greater and the angle BAD less than a right angle.

From D, any point in AD, draw DB perpendicular to AB, and DE perpendicular to AE: from B, draw BC perpendicular to AE and produce DE to meet BF, the perpendicular from B upon DE in F.

By this construction, BC = FE, BF = CE, and the angle BDF is equal to the angle BAC.

$$\text{Also } \frac{BC}{AB} \text{ or } \frac{BF}{BD} = \text{sin. } A.$$

$$\frac{AC}{AB} \text{ or } \frac{DF}{BD} = \text{cos. } A.$$

$$\frac{BD}{AD} = \text{sin. } B.$$

$$\frac{AB}{AD} = \text{cos. } B.$$

$$\frac{DE}{AD} = \text{sin. } (A + B).$$

$$\frac{AE}{AD} = \text{cos. } (A + B).$$

$$\text{Now } \frac{DE}{AD} \left(\text{or sin.} (A + B)\right) = \frac{FE - FD}{AD} = \frac{BC - FD}{AD} = \frac{BC}{AD} - \frac{FD}{AD}.$$

$$\frac{BC}{AD} = \frac{BC}{AB} \times \frac{AB}{AD} = \text{sin. } A \text{ cos. } B.$$

$$\frac{FD}{AD} = \frac{FD}{BD} \times \frac{BD}{AD} = \text{cos. } A \text{ sin. } B.$$

Consequently, sin. $(A + B)$ = sin. A cos. B — cos. A sin. B.

Since the angle A is greater than 90°, cos. A is negative; hence the negative sign of the last term of this result. The equation may be written thus —

$$\text{sin. } (A + B) = \text{sin. } A \text{ cos. } B + (-\text{cos. } A) \times \text{sin. } B.$$

$$\text{Again, } \frac{AE}{AD} \text{ or cos. } (A + B) = \frac{AC + CE}{AD} = \frac{AC + BF}{AD} = \frac{AC}{AD} + \frac{BF}{AD}.$$

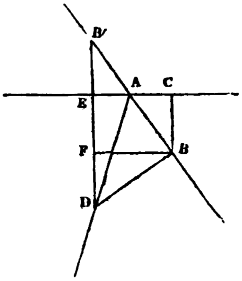
$$\text{But } \frac{AC}{AD} = \frac{AC}{AB} \times \frac{AB}{AD} = \text{cos. } A \text{ cos. } B.$$

$$\text{And } \frac{BF}{AD} = \frac{BF}{BD} \times \frac{BD}{AD} = \text{sin. } A \text{ sin. } B.$$

$$\therefore \text{cos. } (A + B) = \text{cos. } A \text{ cos. } B + \text{sin. } A \text{ sin. } B.$$

In this result the absolute values only of the expressions are considered: but if their relative values also are taken into account, the cosine of the sum of the angles, A, B, (which is greater than 90°), is negative. For the same reason cos. A, and therefore the product cos. A cos. B, are negative. Whence the formula becomes —cos. $(A + B)$ = —cos. A cos. B + sin. A sin. B; or changing the signs of all the terms, cos. $(A + B)$ = cos. A cos. B — sin. A sin. B.

b. Lastly, let each of the angles B′AC or A, B′AD or B be greater than a right angle.



Produce B'A, CA; from D, any point in AD, draw DB perpendicular to AB', and DE perpendicular to AC: also from B draw BC perpendicular to AC and BF perpendicular to DE; and produce DE until it intersects AB in B'.

By this construction the triangles AB'E, ABC, BDF, are equiangular, and BC is equal to FE.

$$\text{Whence, } \frac{\mathrm{BF}}{\mathrm{BD}}=\frac{\mathrm{B'E}}{\mathrm{B'A}}=\frac{\mathrm{BC}}{\mathrm{BA}}=\sin.\ \mathrm{A}.$$

$$\frac{\mathrm{AE}}{\mathrm{AB'}}=\frac{\mathrm{AC}}{\mathrm{AB}}=\frac{\mathrm{FD}}{\mathrm{BD}}=\cos.\ \mathrm{A}.$$

$$\frac{\mathrm{BD}}{\mathrm{AD}}=\sin.\ \mathrm{B}.$$

$$\frac{\mathrm{AB}}{\mathrm{AD}}=\cos.\ \mathrm{B}.$$

$$\frac{\mathrm{ED}}{\mathrm{AD}}=\sin.\ (\mathrm{A}+\mathrm{B}).$$

$$\frac{\mathrm{AE}}{\mathrm{AD}}=\cos.\ (\mathrm{A}+\mathrm{B}).$$

$$\text{Now, } \frac{\mathrm{ED}}{\mathrm{AD}}=\frac{\mathrm{EF}+\mathrm{FD}}{\mathrm{AD}}=\frac{\mathrm{BC}+\mathrm{FD}}{\mathrm{AD}}=\frac{\mathrm{BC}}{\mathrm{AD}}+\frac{\mathrm{FD}}{\mathrm{AD}}.$$

$$\frac{\mathrm{BC}}{\mathrm{AD}}=\frac{\mathrm{BC}}{\mathrm{BA}}\times\frac{\mathrm{BA}}{\mathrm{AD}}=\sin.\ \mathrm{A}\ \cos.\ \mathrm{B}.$$

$$\frac{\mathrm{FD}}{\mathrm{AD}}=\frac{\mathrm{FD}}{\mathrm{BD}}\times\frac{\mathrm{BD}}{\mathrm{AD}}=\cos.\ \mathrm{A}\ \sin.\ \mathrm{B}.$$

$$\therefore \sin.\ (\mathrm{A}+\mathrm{B})=\sin.\ \mathrm{A}\ \cos.\ \mathrm{B}+\cos.\ \mathrm{A}\ \sin.\ \mathrm{B}.$$

It is to be observed, that since the angle A is greater than 90°, cos. A is negative; that since the angle B is greater than 90°, cos. B is negative; and that since the angle A + B is greater than 180°, sin. (A + B) is negative. Whence, taking into consideration the relative values of these expressions, the formula becomes,

$$-\sin.\ (\mathrm{A}+\mathrm{B}) = -\sin.\mathrm{A}\ \cos.\ \mathrm{B}-\cos.\mathrm{A}\ \sin.\mathrm{B}.$$

Or, changing the signs of all the terms,

$$\sin.\ (\mathrm{A}+\mathrm{B})=\sin.\ \mathrm{A}\ \cos.\ \mathrm{B}+\cos.\ \mathrm{A}\ \sin.\ \mathrm{B}.$$

$$\text{Again, } \cos.\ (\mathrm{A}+\mathrm{B}) = \frac{\mathrm{AE}}{\mathrm{AD}}=\frac{\mathrm{CE}-\mathrm{AC}}{\mathrm{AD}}=\frac{\mathrm{BF}-\mathrm{AC}}{\mathrm{AD}}=\frac{\mathrm{BF}}{\mathrm{AD}}-\frac{\mathrm{AC}}{\mathrm{AD}}.$$

$$\frac{\mathrm{BF}}{\mathrm{AD}}=\frac{\mathrm{BF}}{\mathrm{BD}}\times\frac{\mathrm{BD}}{\mathrm{AD}}=\sin.\ \mathrm{A}\ \sin.\ \mathrm{B}.$$

$$\frac{\mathrm{AC}}{\mathrm{AD}}=\frac{\mathrm{AC}}{\mathrm{AB}}\times\frac{\mathrm{AB}}{\mathrm{AD}}=\cos.\ \mathrm{A}\ \cos.\ \mathrm{B}.$$

$$\therefore \cos.\ (\mathrm{A}+\mathrm{B})=\sin.\ \mathrm{A}\ \sin.\ \mathrm{B}-\cos.\ \mathrm{A}\ \cos.\ \mathrm{B}.$$

But since the angle A + B is greater than 180° cos. (A + B) is negative, and since the angles A and B are both greater than 90°, cos. A and cos. B are both negative.

$$\therefore -\cos.\ (\mathrm{A}+\mathrm{B})=\sin.\ \mathrm{A}\ \sin.\ \mathrm{B}-(-\cos.\ \mathrm{A})\times(-\cos.\ \mathrm{B}),$$

$$\text{or}\quad -\cos.\ (\mathrm{A}+\mathrm{B})=\sin.\ \mathrm{A}.\ \sin.\ \mathrm{B}-\cos.\ \mathrm{A}\ \cos.\ \mathrm{B},$$

$$\text{or}\quad \cos.\ (\mathrm{A}+\mathrm{B})=\cos.\ \mathrm{A}\ \cos.\ \mathrm{B}-\sin.\ \mathrm{A}\ \sin.\ \mathrm{B}.$$

The preceding discussion may suffice to show that if the relative values of sin. A, cos. A, sin. B, cos. B, sin. (A ± B), cos. (A ± B) are considered, the formulæ of this Article are general.

32. From the formulæ of Art. 31. may be deduced others expressing $\sin.(A \pm B \pm C)$, $\cos.(A \pm B \pm C)$, in terms of the sines and cosines of the angles A, B, C.

For $A \pm B \pm C$ may be put under the form $(A \pm B) \pm C$;

$$\therefore \ \text{Sin}\,(A \pm B \pm C) = \text{Sin.}\ \{(A \pm B) \pm C\}.$$

$$\text{But sin.}\ \{(A \pm B) \pm C\} = \sin.(A \pm B)\cos.C \quad \pm \quad \cos.(A \pm B)\sin.C$$

$$= (\sin.A\cos.B \pm \cos.A\sin.B)\cos.C \pm (\cos.A\cos.B \mp \sin.A\sin.B)\sin.C$$

$$\text{or } \sin.(A \pm B \pm C) = \sin.A\cos.B\cos.C \pm \sin.B\cos.A\cos.C \pm \sin.C\cos.A\cos.B - \sin.A\sin.B\sin.C.$$

In like manner, $\cos.(A \pm B \pm C) = \cos.\{(A \pm B) \pm C\}$

$$\text{Cos.}\ \{(A \pm B) \pm C\} = \cos.(A \pm B)\cos.C \quad \mp \sin.(A \pm B)\sin.C$$

$$= (\cos.A\cos.B \mp \sin.A\sin.B)\cos.C \mp (\sin.A\cos.B \pm \cos.A\sin.B)\sin.C$$

$$\text{or } \cos.(A \pm B \pm C) = \cos.A\cos.B\cos.C \mp \sin.A\sin.B\cos.C \mp \sin.A\cos.B\sin.C - \cos.A\sin.B\sin.C.$$

a. If A, B, C, are the three angles of a triangle, $\sin.(A + B + C) = \sin.\ 180° = 0$; and the first formula of this article becomes

$$0 = \sin.A\cos.B\cos.C + \sin.B\cos.A\cos.C + \sin.C\cos.A\cos.B - \sin.A\sin.B\sin.C$$

$$\text{or, } \sin.A\sin.B\sin.C = \sin.A\cos.B\cos.C + \sin.B\cos.A\cos.C + \sin.C\cos.A\cos.B.$$

33. If $A = B = C$, $\sin.^3 A = 3\sin.A\cos.^2 A$ or $\sin.^2 A = 3\cos.^2 A$.

Under this hypothesis, the triangle whose angles are A, B, C, is equilateral: therefore each angle is equal to $\frac{180°}{3} = 60°$.

Replacing A by this particular value, $\sin.^2 60° = 3\cos.^2 60°$.

$$\text{But } \cos.^2 60° = 1 - \sin.^2 60°$$

$$\therefore \ \text{Sin.}^2 60° = 3\,(1 - \sin.^2 60°) = 3 - 3\sin.^2 60$$

$$\text{or, } 4\sin.^2 60° = 3;\ \sin.^2 60 = \frac{3}{4}$$

$$\text{And, } \sin.\ 60 = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

a. Again, since $3\cos.^2 60° = \sin.^2 60°$, and $\sin.^2 60° = 1 - \cos.^2 60°$

$$3\cos.^2 60° = 1 - \cos.^2 60°, \text{ or } 4\cos.^2 60° = 1$$

$$\therefore \ \text{Cos.}^2 60° = \frac{1}{4}, \text{ and } \cos.60° = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

b. Sin. $60° = \cos.\ 30°$, and $\cos.\ 60° = \sin.\ 30°$. (Art. 17.).

$$\text{But, } \sin.\ 60° = \frac{\sqrt{3}}{2} \text{ and } \cos.\ 60° = \frac{1}{2}$$

$$\therefore \ \text{Cos.}\ 30° = \frac{\sqrt{3}}{2}, \text{ and } \sin.\ 30° = \frac{1}{2}.$$

c. Since $\tan.A = \frac{\sin.A}{\cos.A}$, $\tan.\ 30° = \frac{\sin.30°}{\cos.30} = \frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$;

$$\tan.\ 60° = \frac{\sin.\ 60°}{\cos.\ 60} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}.$$

d. $\sin. 45° = \cos. (90° - 45°) = \cos. 45°$. (Art. 17.).

$\therefore \sin.^2 45° + \cos.^2 45°$, or, $\sin.^2 45° + \sin.^2 45° = 1$. (Art. 19. *d.*)

$$\therefore\ 2 \sin.^2 45° = 1, \text{ and } \sin. 45° = \frac{1}{\sqrt{2}}.$$

Also, since $\cos. 45° = \sin. 45°$ $\therefore$ $\cos. 45° = \frac{1}{\sqrt{2}}$.

Hence $\tan. 45° = \sin. 45° \div \cos. 45° = \frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}} = 1$.

e. The values of the sines and cosines of 45° and 30° being determined, the values of the sines and cosines of $45° \pm 30°$, or 75° and 15°, can be found by means of the formulæ of Art. 31. For by substituting the values, $\sin. (45° \pm 30°) = \sin. 45° \cos. 30 \pm \cos. 45° \sin. 30$, gives

$$\sin. 75° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\sin. 15° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

f. The numerical values of cos. 75° and cos. 15° can be deduced in the same manner from the formula, $\cos. (45° \pm 30°) = \cos. 45° \cos. 30° \mp \sin. 45° \sin. 30°$, or more easily from the expressions

$$\cos. 75° = \sin. (90° - 75°) = \sin. 15° ; \cos. 15° = \sin. (90° - 15°) = \sin. 75°.$$

$$\therefore\ \cos. 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}, \text{ and } \cos. 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

34. Expressions which give the sines and cosines of the multiple angles $2A$, $3A$, $4A$, &c. in terms of the sine and cosine of the angle A, are obtained from the formulæ of Art. 31., by making successively $B = A$, $B = 2A$, $B = 3A$, &c., as follows: —

In the formula, $\sin. (A + B) = \sin. A \cos. B + \cos A \sin. B$, let $B = A$:

$$\therefore\ \sin. (A + A) = \sin. A \cos. A + \cos. A \sin. A$$

$$\text{or, } \sin. 2A = 2 \sin. A \cos. A$$

Substituting $\frac{A'}{2}$ for A, and therefore A' for $2A$, this formula becomes

$$\sin. A' = 2 \sin. \frac{A'}{2} \cos. \frac{A'}{2}.$$

a. Similarly, in the formula $\cos. (A + B) = \cos. A \cos. B - \sin. A \sin. B$, let $B = A$:

$$\therefore \cos. (A + A) = \cos. A \cos. A - \sin. A \sin. A$$

$$\text{or, } \cos. 2A = \cos.^2 A - \sin.^2 A.$$

If $\frac{A'}{2}$ and A' are substituted for A and $2A$, this formula becomes

$$\cos. A' = \cos.^2 \frac{A'}{2} - \sin.^2 \frac{A'}{2}.$$

b. Making $B = 2A$, $\sin. (A + 2A) = \sin. A \cos. 2A + \cos. A \sin. 2A$.

Substituting $\cos.^2 A - \sin.^2 A$ for $\cos. 2A$, and $2 \sin. A \cos. A$ for $\sin. 2A$,

$$\sin. (A + 2A) = \sin. A (\cos.^2 A - \sin.^2 A) + \cos. A \times 2 \sin. A \cos. A$$

$$\text{or, } \sin. 3A = \sin. A \cos.^2 A - \sin.^3 A + 2 \sin. A \cos.^2 A$$

$$\therefore \sin. 3A = 3 \sin. A \cos.^2 A - \sin.^3 A.$$

c. In like manner, $\cos. (A + 2A) = \cos. A \cos. 2A - \sin. A \sin. 2A$.

$$\text{or, } \cos. 3A = \cos. A (\cos.^2 A - \sin.^2 A) - \sin. A \times 2 \sin. A \cos. A$$
$$\text{or, } \cos. 3A = \cos.^3 A - \cos. A \sin.^2 A - 2 \cos. A \sin.^2 A$$
$$\text{or, } \cos. 3A = \cos.^3 A - 3 \cos. A \sin.^2 A.$$

d. If $\frac{A''}{3}$ and A'' are substituted for A and $3A$, formulæ *b*, *c*, become

$$\sin. A'' = 3 \sin. \frac{A''}{3} \cos.^2 \frac{A''}{3} - \sin.^3 \frac{A''}{3}.$$

$$\cos. A'' = \cos.^3 \frac{A''}{3} - 3 \cos. \frac{A''}{3} \sin.^2 \frac{A''}{3}.$$

e. The expressions for $\sin. 3A$, $\cos. 3A$ are immediately deducible from the formulæ of Article 32, by making $A = B = C$: thus:

$\sin. (A+B+C) = \sin. A \cos. B \cos. C + \sin. B \cos. A \cos. C + \sin. C \cos. A \cos. B - \sin. A \sin. B \sin. C$, becomes
$\sin. (A+A+A) = \sin. A \cos. A \cos. A + \sin. A \cos. A \cos. A + \sin. A \cos. A \cos. A - \sin. A \sin. A \sin. A$

$$\text{or, } \sin. 3A = \sin. A \cos.^2 A + \sin. A \cos.^2 A + \sin. A \cos.^2 A - \sin.^3 A,$$
$$\text{or, } \sin. 3A = 3 \sin. A \cos.^2 A - \sin.^3 A:$$

And $\cos. (A + B + C)$ becomes $\cos. 3A = \cos.^3 A - \cos. A \sin.^2 A - \cos. A \sin.^2 A - \cos. A \sin.^2 A$,

$$\text{or, } \cos. 3A = \cos.^3 A - 3 \cos. A \sin.^2 A.$$

Corresponding expressions for $\sin. 4A$, $\cos. 4A$, $\sin. 5A$, $\cos. 5A$ .... can be in the same manner deduced from the formulæ of Articles 31. and 32., or more readily from the formula of Demoivre, which is given in Art. 43.

35. To find formulæ which shall express the sine and cosine of half an angle in terms of the sine and cosine of the whole angle.

$$\text{Since } \cos.^2 A + \sin.^2 A = 1, \text{ (Art. 19. } d.)$$
$$\text{and } \cos.^2 A - \sin.^2 A = \cos. 2A, \text{ (Art. 34. } a.)$$

$$\text{by addit. } 2\cos.^2 A = 1 + \cos. 2A \;\therefore\; \cos. A = \pm\sqrt{\frac{1+\cos. 2A}{2}}$$

$$\left.\begin{array}{l}\text{and by}\\ \text{subtract.}\end{array}\right\} 2\sin.^2 A = 1 - \cos. 2A \;\therefore\; \sin. A = \pm\sqrt{\frac{1-\cos. 2A}{2}}$$

The only condition which the angles A, 2A, are required to satisfy is, that the first shall be equal to one half of the second: if, therefore, the halves of A and 2A are taken, the preceding results are still true; whence

$$2\cos.^2\frac{A}{2} = 1 + \cos. A \text{ and } \cos.\frac{A}{2} \text{ or } \cos.\tfrac{1}{2}A = \pm\sqrt{\frac{1+\cos. A}{2}}$$

$$2\sin.^2\frac{A}{2} = 1 - \cos. A, \quad \text{and} \quad \sin.\tfrac{1}{2}A = \pm\sqrt{\frac{1-\cos. A}{2}}$$

a. These formulæ can be transformed into others, expressed in terms of sin. A.

For $\sqrt{\frac{1+\cos. A}{2}} = \sqrt{\tfrac{1}{2}(1+\cos. A)}$, $\sqrt{\frac{1-\cos. A}{2}} = \sqrt{\tfrac{1}{2}(1-\cos. A)}$ and $\cos. A = \pm\sqrt{1-\sin.^2 A}$.

$$\therefore \cos.\tfrac{1}{2}A = \pm\sqrt{\tfrac{1}{2}(1\pm\sqrt{1-\sin.^2 A})}$$
$$\sin.\tfrac{1}{2}A = \pm\sqrt{\tfrac{1}{2}(1\mp\sqrt{1-\sin.^2 A})}$$

36. The sum and difference of the sine and cosine of the same angle can be obtained from the formulæ of Art. 35., or, in a less complicated form, as follows: —

$$\text{since } \cos.^2 A + \sin.^2 A = 1, \text{ (Art. 19. } d.)$$
$$\text{and } 2\sin. A \cos. A = \sin. 2A, \text{ (Art. 34.)}$$

$$\therefore \text{ by addition, } \cos.^2 A + 2\sin. A\cos. A + \sin.^2 A = 1 + \sin. 2A:$$
$$\text{and by subtraction, } \cos.^2 A - 2\sin. A\cos. A + \sin.^2 A = 1 - \sin. 2A.$$

$$\text{But } \cos.^2 A + 2\sin. A \cos. A + \sin.^2 A = (\cos. A + \sin. A)^2$$
$$\text{and } \cos.^2 A - 2\sin. A \cos. A + \sin.^2 A = (\cos. A - \sin. A)^2$$

$$\therefore (\cos. A + \sin. A)^2 = 1 + \sin. 2A; \text{ and } \cos. A + \sin. A = \pm\sqrt{1+\sin. 2A}$$
$$(\cos. A - \sin. A)^2 = 1 - \sin. 2A; \text{ and } \cos. A - \sin. A = \pm\sqrt{1-\sin. 2A}$$

37. Formulæ expressing the tangents of the sum and difference of two or more angles in terms of the tangents of the single angles, can be deduced from the corresponding expressions of the sines and cosines of these angles.

For since the tangent of any angle is equal to the sine of that angle divided by the cosine, (Art. 19. *c.*)

$$\text{Tan. }(\text{A} \pm \text{B}) = \frac{\text{sin. }(\text{A} \pm \text{B})}{\text{cos. }(\text{A} \pm \text{B})} = \frac{\text{sin. A cos. B} \pm \text{cos. A sin. B}}{\text{cos. A cos. B} \mp \text{sin. A sin. B}}.$$

This formula is reduced to another of the same value, expressed in terms of the tangents of A and B, by dividing every term of the numerator and denominator of its second number by cos. A cos. B.

For, if the division is made, the formula becomes

$$\text{tan. }(\text{A} \pm \text{B}) = \frac{\dfrac{\text{sin. A}}{\text{cos. A}} \pm \dfrac{\text{sin. B}}{\text{cos. B}}}{1 \mp \dfrac{\text{sin. A}}{\text{cos. A}} \cdot \dfrac{\text{sin. B}}{\text{cos. B}}}$$

And if tan. A, tan. B are substituted for their equivalents $\dfrac{\text{sin. A}}{\text{cos. A}}, \dfrac{\text{sin. B}}{\text{cos. B}}.$

$$\text{Tan. }(\text{A.} \pm \text{B}) = \frac{\text{tan. A} \pm \text{tan. B}}{1 \mp \text{tan. A. tan. B}}.$$

a. The expression of tan. (A ± B ± C), in terms of the tangents of the angles A, B, C, can be deduced from that of tan. (A ± B); or from those of sin. (A ± B ± C), cos. (A ± B ± C) in Article 32.

Taking only the sum A + B + C, adopting the second method, and dividing every term of the numerator and denominator of the result by cos. A cos. B cos. C,

$$\text{Tan. }(\text{A} + \text{B} + \text{C}) = \frac{\dfrac{\text{sin. A}}{\text{cos A}} + \dfrac{\text{sin. B}}{\text{cos. B}} + \dfrac{\text{sin. C}}{\text{cos. C}} - \dfrac{\text{sin. A sin. B sin. C}}{\text{cos. A cos. B cos. C}}}{1 - \dfrac{\text{sin. A sin. B}}{\text{cos. A cos. B}} - \dfrac{\text{sin. A sin. C}}{\text{cos. A cos. C}} - \dfrac{\text{sin. B sin. C}}{\text{cos. B cos. C}}}$$

$$\text{or tan. }(\text{A} + \text{B} + \text{C}) = \frac{\text{tan. A} + \text{tan. B} + \text{tan. C} - \text{tan. A tan. B tan. C}}{1 - \text{tan. A tan. B} - \text{tan. A tan. C} - \text{tan. B tan. C}}$$

In the same manner, tan. (A ± B ± C ± D . . .) may be found in terms of the tangents of the angles A, B, C, D, . . .

b. If A, B, C, are the three angles of a triangle, tan. (A + B + C) = tan. 180° = 0: whence the numerator of the second member of the equation 37 *a*, must be equal to 0: that is, tan. A + tan. B + tan. C − tan. A tan. B tan. C = 0,

$$\therefore \text{ tan. A} + \text{tan. B} + \text{tan. C} = \text{tan. A tan. B tan. C}.$$

38. To express tan. 2 A, tan. 3 A, tan. $\frac{1}{2}$ A, in terms of tan. A —in the formula of Article 37., which is,

$$\tan.(\text{A}+\text{B})=\frac{\tan.\text{A}+\tan.\text{B}}{1-\tan.\text{A}\tan.\text{B}},\ \text{let B}=\text{A}$$

$$\therefore\ \tan.2\text{A}=\frac{2\tan.\text{A}}{1-\tan.^2\text{A}}.$$

a. Similarly, if in the expression,

$$\tan.(\text{A}+\text{B}+\text{C})=\frac{\tan.\text{A}+\tan.\text{B}+\tan.\text{C}-\tan.\text{A},\tan.\text{B},\tan.\text{C}}{1-\tan.\text{A}\tan.\text{B}-\tan.\text{A}\tan.\text{C}-\tan\text{B}\tan.\text{C}},\ \text{A}=\text{B}=\text{C}$$

$$\tan.3\text{A}=\frac{3\tan.\text{A}-\tan.^3\text{A}}{1-3\tan.^2\text{A}}.$$

b. To express tan. $\frac{1}{2}$A in terms of tan. A: in the formula $\tan.2\text{A}=\frac{2\tan.\text{A}}{1-\tan.^2\text{A}}$ let 2A be replaced by A, and A by $\frac{1}{2}$A, then

$$\tan.\text{A}=\frac{2\tan.\frac{1}{2}\text{A}}{1-\tan.^2\frac{1}{2}\text{A}}$$

$$\therefore\ \tan.\text{A}-\tan.\text{A}\tan.^2\frac{\text{A}}{2}=2\tan.\frac{\text{A}}{2}$$

$$\text{or, }\tan.\text{A}\tan.^2\frac{\text{A}}{2}+2\tan.\frac{\text{A}}{2}=\tan.\text{A},$$

$$\text{or, }\tan.^2\frac{\text{A}}{2}+\frac{2}{\tan.\text{A}}\tan.\frac{\text{A}}{2}=1.$$

Resolving this quadratic equation, it is found that

$$\tan.\frac{\text{A}}{2}=-\frac{1}{\tan.\text{A}}\pm\frac{1}{\tan.\text{A}}\sqrt{1+\tan.^2\text{A}}.$$

c. The value of tan. $\frac{1}{2}$A can be also assigned in terms of the sine and cosine of A: For $\tan.\frac{1}{2}\text{A}=\frac{\sin.\frac{1}{2}\text{A}}{\cos.\frac{1}{2}\text{A}}$, and, by Article 35.,

$$\sin.\tfrac{1}{2}\text{A}=\sqrt{\frac{1-\cos.\text{A}}{2}},\ \cos.\tfrac{1}{2}\text{A}=\sqrt{\frac{1+\cos.\text{A}}{2}}$$

$$\therefore\tan.\tfrac{1}{2}\text{A}=\sqrt{\frac{1-\cos.\text{A}}{2}}\div\sqrt{\frac{1+\cos.\text{A}}{2}}=\sqrt{\frac{1-\cos.\text{A}}{1+\cos.\text{A}}}.$$

If, to simplify this expression, the terms of the quantity under the radical sign are multiplied by 1 + cos. A.

$$\tan.\tfrac{1}{2}\text{A}=\sqrt{\frac{1-\cos.^2\text{A}}{(1+\cos.\text{A})^2}}=\sqrt{\frac{\sin.^2\text{A}}{(1+\cos.\text{A})^2}}=\frac{\sin.\text{A}}{1+\cos.\text{A}}.$$

Or, if the terms of the radical are multiplied by 1 − cos. A,

$$\tan.\tfrac{1}{2}\text{A}=\sqrt{\frac{(1-\cos.\text{A})^2}{1-\cos.^2\text{A}}}=\sqrt{\frac{(1-\cos.\text{A})^2}{\sin.^2\text{A}}}=\frac{1-\cos.\text{A}}{\sin.\text{A}}.$$

39. Since $\sin.(A+B)=\sin.A\cos.B+\cos.A\sin.B$. (Art. 31.)
and $\sin.(A-B)=\sin.A\cos.B-\cos.A\sin.B$.

by addition, $\sin.(A+B)+\sin.(A-B)=2\sin.A\cos.B$ 1

and by subtraction, $\sin.(A+B)-\sin.(A-B)=2\cos.A\sin.B$ 2

Again, since $\cos.(A-B)=\cos.A\cos.B+\sin.A\sin.B$,
and $\cos.(A+B)=\cos.A\cos.B-\sin.A\sin.B$,

by addition, $\cos.(A-B)+\cos.(A+B)=2\cos.A\cos.B$ 3

and by subtraction, $\cos.(A-B)-\cos.(A+B)=2\sin.A\sin.B$ 4

If the first formula of this article is divided by the second

$$\frac{\sin.(A+B)+\sin.(A-B)}{\sin.(A+B)-\sin.(A-B)}=\frac{2\sin.A\cos.B}{2\cos.A\sin.B}=\frac{\sin.A}{\cos.A}\cdot\frac{\cos.B}{\sin.B}=\tan.A\cdot\frac{1}{\tan.B}=\frac{\tan.A}{\tan.B}$$

And if the third is divided by the fourth,

$$\frac{\cos.(A-B)+\cos.(A+B)}{\cos.(A-B)-\cos.(A+B)}=\frac{2\cos.A\cos.B}{2\sin.A\sin.B}=\frac{\cos.A}{\sin.A}\cdot\frac{\cos.B}{\sin.B}=\cot.A\cot.B.$$

40. If A, B, denote two unequal angles, of which A is the greater: then $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ denote the half sum and half difference of these angles.

If the half difference is added to the half sum, the result, which is $\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$, or $\frac{A}{2}+\frac{B}{2}+\frac{A}{2}-\frac{B}{2}$, or $\frac{A}{2}+\frac{A}{2}$, or A, is equal to the greater angle A:

And if the half difference is subtracted from the half sum, the remainder, which is $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)$, or $\frac{A}{2}+\frac{B}{2}-\frac{A}{2}+\frac{B}{2}$ or $\frac{B}{2}+\frac{B}{2}$, or B, is equal to the less angle B.

Now since $A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$; $\sin.A=\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)+\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$
And since $B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B)$; $\sin.B=\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)-\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$

$\therefore$ by addition, $\sin.A+\sin.B=2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)$ . . 1.
And, by subtraction, $\sin.A-\sin.B=2\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$. 2.

Dividing formula 1 by formula 2,

$$\frac{\sin.A+\sin.B}{\sin.A-\sin.B}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}{2\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)}=\frac{\sin.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A+B)}\cdot\frac{\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A-B)}=\frac{\tan.\frac{1}{2}(A+B)}{\tan.\frac{1}{2}(A-B)}$$

a. Since $A=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)$; $\cos.A=\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)-\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$
and since $B=\frac{1}{2}(A+B)-\frac{1}{2}(A-B)$; $\cos.B=\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)+\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$

Therefore, by addition, $\cos.A+\cos.B=2\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)$ . . 1.
And, by subtraction, $\cos.B-\cos.A=2\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)$ . . 2.

Dividing formula 1 by formula 2,

$$\frac{\cos.A+\cos.B}{\cos.B-\cos.A}=\frac{\cos.\frac{1}{2}(A+B)}{\sin.\frac{1}{2}(A+B)}\cdot\frac{\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A-B)}=\cot.\tfrac{1}{2}(A+B)\cot.\tfrac{1}{2}(A-B).$$

b. Since $\sin. A + \sin. B = 2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)$

And $\cos. A + \cos. B = 2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)$

$$\therefore \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. \frac{1}{2}(A+B)}{\cos. \frac{1}{2}(A+B)} = \tan. \tfrac{1}{2}(A+B)$$

In the same manner it is found that

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \tan. \tfrac{1}{2}(A-B)$$

and that $\dfrac{\sin. A \pm \sin. B}{\cos. B - \cos. A} = \cot. \tfrac{1}{2}(A \mp B)$.

The formulæ of Articles 39 and 40 are employed to change the sums and differences of certain functions of angles into the products or quotients of either the same or of different functions of these angles, and by consequence to adapt to logarithmic calculation the numerical values of these functions, or of the functions by which they are replaced.

41. Let n be a whole number, A any angle, and let A, B, in the formulæ of Article 31. be replaced by $(n-1)$ A and A: then

$$\sin. ((n-1)A + A) = \sin. (n-1)A \cos. A + \cos. (n-1)A \sin. A$$

And $\sin. \{(n-1)A - A\} = \sin. (n-1)A \cos. A - \cos. (n-1)A \sin. A.$

Therefore, by addition, $\sin. \{(n-1)A + A\} + \sin. \{(n-1)A - A\} = 2 \sin. (n-1)A \cos. A.$

and by subtraction, $\sin. \{(n-1)A + A\} - \sin. \{(n-1)A - A\} = 2 \cos. (n-1)A \sin. A$

Or, since $(n-1)A + A = nA$; $(n-1)A - A = (n-2)A$,

$$\sin. nA + \sin. (n-2)A = 2 \sin. (n-1)A \cos. A \quad . \quad . \quad 1.$$

$$\sin. nA - \sin. (n-2)A = 2 \cos. (n-1)A \sin. A \quad . \quad . \quad 2.$$

Similarly, $\cos. \{(n-1)A + A\} = \cos. (n-1)A \cos. A - \sin. (n-1)A \sin. A,$

$\cos. \{(n-1)A - A\} = \cos. (n-1)A \cos. A + \sin. (n-1)A \sin. A.$

Therefore, by addition, $\cos. nA + \cos. (n-2)A = 2 \cos. (n-1)A \cos. A$ ..3.

And, by subtraction, $\cos. (n-2)A - \cos. nA = 2 \sin. (n-1)A \sin. A$ ..4.

By means of these formulæ, the sines and cosines of the multiple angles, 2A, 3A, 4A . . . . can be expressed in terms of the powers of the sine and cosine of the angle A.

a. For, if in formula 1., $n=2$, $\sin. 2A = 2 \sin. A \cos. A.$

If $n=3$, $\sin. 3A + \sin. A = 2 \times \sin. 2A \times \cos. A = 2 \times 2 \sin. A \cos. A \times \cos. A = 4 \sin. A \cos.^2 A$

$$\therefore \sin. 3A = 4 \sin. A \cos.^2 A - \sin. A:$$

or substituting $1 - \sin.^2 A$ for $\cos.^2 A$;

$$\sin. 3A = 4 \sin. A (1 - \sin.^2 A) - \sin. A = 4 \sin. A - 4 \sin.^3 A - \sin. A = 3 \sin. A - 4 \sin.^3 A.$$

If $n=4$, $\sin. 4A + \sin. 2A = 2 \sin. 3A \cos. A.$

If, in this expression, $\sin. 2A$ and $\sin. 3A$ are replaced by $2 \sin. A \cos. A$ and $4 \sin. A \cos.^2 A - \sin. A$

the result is $\sin. 4A = 8 \sin. A \cos.^3 A - 4 \sin. A \cos. A,$

And if $\sin. 3A$ is replaced by $3 \sin. A - 4 \sin.^3 A$, the result is

$$\sin. 4A = 4 \sin. A \cos. A - 8 \sin.^3 A \cos. A.$$

This expression may be deduced from the other by putting $8 \sin. A \cos.^3 A$ or $8 \sin. A \cos. A \cos.^2 A$ under the form $8 \sin. A \cos. A (1 - \sin.^2 A)$, and making the necessary reductions.

If $n=5$; $\sin. 5A = 16 \sin. A \cos.^4 A - 12 \sin. A \cos.^2 A + \sin. A.$

Again,
if in formula 3., $n=2$; $\cos.2A+\cos.(2-2)A=2\cos.A\cos.A=2\cos.^2A$,

$$\text{or, since } \cos.0A=\cos.0=1,\ \cos.2A+1=2\cos.^2A,$$

$$\cos.2A=2\cos.^2A-1.$$

$$\text{If } n=3,\ \cos.3A+\cos.A=2\times\cos.2A\times\cos A=2(2\cos.^2A-1)\cos.A=4\cos.^3A-2\cos.A$$

$$\therefore \cos.3A=4\cos.^3A-3\cos.A$$ [*]

$$\text{If } n=4,\ \cos.4A+\cos.2A=2\cos.3A\cos.A,$$

$$\text{or } \cos.4A+2\cos.^2A-1=2(4\cos.^3A-3\cos.A)\cos.A,$$

$$\text{or } \cos.4A+2\cos.^2A-1=8\cos.^4A-6\cos.^2A,$$

$$\therefore \cos.4A=8\cos.^4A-8\cos.^2A+1.$$

. . . .

By means of formula 4., cos. *n* A may be expressed in terms of the powers of sin. A.

42. If $(n-1)A$ and A are substituted for A, B, in the formula,

$$\tan.(A+B)=\frac{\tan.A+\tan.B}{1-\tan.A\tan.B}.$$

$$\tan.\{(n-1)A+A\}=\frac{\tan.(n-1)A+\tan.A}{1-\tan.(n-1)A\tan.A}.$$

$$\text{or, } \tan.nA=\frac{\tan.(n-1)A+\tan.A}{1-\tan.(n-1)A\tan.A}.$$

If, in this expression, $n=2$, $\tan.2A=\dfrac{\tan.A+\tan.A}{1-\tan.A\tan.A}=\dfrac{2\tan.A}{1-\tan.^2A}$.

$$\text{If } n=3,\ \tan.3A=\frac{\tan.2A+\tan.A}{1-\tan.2A\tan.A}=\frac{\dfrac{2\tan.A}{1-\tan.^2A}+\tan.A}{1-\dfrac{2\tan.A}{1-\tan.^2A}\times\tan.A}$$

$$\text{or, } \tan.3A=\frac{2\tan.A+\tan.A-\tan.^3A}{1-\tan.^2A-2\tan.^2A}.$$

$$\text{or, } \tan.3A=\frac{3\tan.A-\tan.^3A}{1-3\tan.^2A}.$$

$$\text{Similarly, if } n=4,\ \tan.4A=\frac{4\tan.A-4\tan.^3A}{1-6\tan.^2A+\tan.^4A}$$

$$\text{If } n=5,\ \tan.5A=\frac{5\tan.A-10\tan.^3A+\tan.^5A}{1-10\tan.^2A+5\tan.^4A}.$$

. . . .

43. Let ϕ and θ denote two arcs which measure the angles A, B, respectively: then, if the expression $\cos.\phi+\sqrt{-1}\sin.\phi$ is multiplied by $\cos.\theta+\sqrt{-1}\sin.\theta$, the product is

$$\cos.\phi\cos.\theta-\sin.\phi\sin.\theta+\sqrt{-1}(\sin.\phi\cos.\theta+\cos.\phi\sin.\theta).$$

$$\left.\begin{array}{l}\text{But } \cos.\phi\cos.\theta-\sin.\phi\sin.\theta=\cos.(\phi+\theta)\\ \text{and } \sin.\phi\cos.\theta+\cos.\phi\sin.\theta=\sin.(\phi+\theta)\end{array}\right\}\ \text{(Art. 31.)}$$

$$\therefore(\cos.\phi+\sqrt{-1}\sin.\phi)(\cos.\theta+\sqrt{-1}\sin.\theta)=\cos.(\phi+\theta)+\sqrt{-1}\sin.(\phi+\theta).$$

Hence, it appears that if two expressions of the form cos. $\varphi + \sqrt{-1}$ sin. φ are multiplied together, the product is a similar expression in which the two arcs are added together. To multiply the product by a third factor of the same form it will be sufficient, therefore, to add the third arc to the two other arcs, and so on, whatever be the number of factors.

---

[*] For the application of this formula to the solution of a cubic equation, see Appendix, Art. 9.

Consequently, if each factor is equal to cos. $\varphi + \sqrt{-1}$ sin. φ, and the number of factor is n,

$$(\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^n = \text{cos. } n\varphi + \sqrt{-1} \text{ sin. } n\varphi \quad . \; . \; . \; 1$$

Substituting $\frac{\varphi}{n}$ for φ, formula 1 becomes

$$\left(\text{cos. } \frac{\varphi}{n} + \sqrt{-1} \text{ sin. } \frac{\varphi}{n}\right)^n = \text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi.$$

Extracting the nth root of both members of this equation,

$$\text{cos. } \frac{\varphi}{n} + \sqrt{-1} \text{ sin. } \frac{\varphi}{n} = (\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^{\frac{1}{n}},$$

or $(\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^{\frac{1}{n}} = \text{cos. } \frac{\varphi}{n} + \sqrt{-1} \text{ sin. } \frac{\varphi}{n} \quad . \; . \; . \; 2$

Again,

since $(\text{cos. } \phi + \sqrt{-1} \text{ sin. } \phi)^m = \text{cos. } m\phi + \sqrt{-1} \text{ sin. } m\phi$. (Form. 1.)

and $(\text{cos. } \phi + \sqrt{-1} \text{ sin. } \phi)^{\frac{1}{n}} = \text{cos. } \frac{\phi}{n} + \sqrt{-1} \text{ sin. } \frac{\phi}{n}$. (Form. 2.)

$$\therefore (\text{cos. } \phi + \sqrt{-1} \text{ sin. } \phi)^{\frac{m}{n}} = \text{cos. } \frac{m}{n}\phi + \sqrt{-1} \text{ sin } \frac{m}{n}\phi \quad . \; . \; . \; . \; 3.$$

Lastly, if cos. $n\phi + \sqrt{-1}$ sin. $n\varphi$ is multiplied by cos. $n\varphi - \sqrt{-1}$ sin. $n\varphi$, the product is

$$\text{cos.}^2 \, n\varphi + \text{sin.}^2 \, n\varphi.$$

But $\text{cos.}^2 \, n\varphi + \text{sin.}^2 \, n\varphi = 1$. (Art. 19. *d.*)

$$\therefore \; 1 = (\text{cos. } n\varphi + \sqrt{-1} \text{ sin. } n\varphi)(\text{cos. } n\varphi - \sqrt{-1} \text{ sin. } n\varphi).$$

$$\text{And } \frac{1}{\text{cos. } n\varphi + \sqrt{-1} \text{ sin. } n\varphi} = \text{cos. } n\varphi - \sqrt{-1} \text{ sin. } n\varphi$$

But $\text{cos. } n\phi + \sqrt{-1} \text{ sin. } n\phi = (\text{cos. } \phi + \sqrt{-1} \text{ sin. } \phi)^n$. (Form. 1.)

$$\therefore \frac{1}{(\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^n} = \text{cos. } n\varphi - \sqrt{-1} \text{ sin. } n\varphi$$

$$\text{or } (\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^{-} = (\text{cos. } n\varphi - \sqrt{-1} \text{ sin. } n\varphi$$

or, since cos. $n\phi$ = cos. $(-n\phi)$, and sin. $n\phi$ = $-$sin. $(-n\phi)$. (Arts. 24. *a.* 18. *e.*)

$$(\text{cos. } \varphi + \sqrt{-1} \text{ sin. } \varphi)^{-n} = \text{cos. } (-n\varphi) + \sqrt{-1} \text{ sin. } (-n\varphi) \; . \; . \; 4$$

Consequently the formula, $(\cos. \varphi + \sqrt{-1} \sin. \varphi)^n = \cos. n\varphi + \sqrt{-1} \sin. n\varphi$ is true for any value of n, whether positive or negative. This is the formula of De Moivre.

a. Formula 2., $(\cos. \phi + \sqrt{-1} \sin. \phi)^{\frac{1}{n}} = \cos. \frac{\phi}{n} + \sqrt{-1} \sin. \frac{\phi}{n}$,

or $\sqrt[n]{\cos. \phi + \sqrt{-1} \sin. \phi} = \cos. \frac{\phi}{n} + \sqrt{-1} \sin. \frac{\phi}{n}$,

being a radical, whose index n is integer and positive, ought to have n different values (Alg. Art. 165.), whilst the second member has only one apparent value. By the substitution, however, for φ of all the arcs which have the same sine and cosine as ϕ, the number of values of the second member is rendered equal to that of the first member, viz. n. The general expression of these arcs is $m \times 2\pi + \phi$ or $2m\pi + \phi$ (Art. 30. *m.*): and the second member of formula 2. by the substitution of this expression for ϕ, becomes

$$\cos. \frac{2m\pi + \phi}{n} + \sqrt{-1} \sin. \frac{2m\pi + \varphi}{n}$$

This expression has precisely the same values as $\sqrt[n]{\cos. \varphi + \sqrt{-1} \sin. \varphi}$. For since n is a whole number, $\left(\cos. \frac{2m\pi + \phi}{n} + \sqrt{-1} \sin. \frac{2m\pi + \varphi}{n}\right)^n = \cos. (2m\pi + \varphi) + \sqrt{-1} \sin. (2m\pi + \varphi)$ and suppressing the circumferences, $2m\pi$, which have no effect on the value,

$$\left(\cos. \frac{2m\pi + \varphi}{n} + \sqrt{-1} \sin. \frac{2m\pi + \varphi}{n}\right)^n = \cos. \varphi + \sqrt{-1} \sin. \varphi = \left(\sqrt[n]{\cos. \varphi + \sqrt{-1} \sin. \varphi}\right)^n$$

Also the different values of $\cos. \frac{2m\pi + \varphi}{n} + \sqrt{-1} \sin. \frac{2m\pi + \varphi}{n}$ are in number n.

For if $m = 0$, the original value, $\cos. \frac{\phi}{n} \sqrt{-1} \sin. \frac{\varphi}{n}$ is obtained.

If $m = 1$, the value is $\cos. \frac{1 \times 2\pi + \varphi}{n} + \sqrt{-1} \sin. \frac{1 \times 2\pi + \varphi}{n}$;

$m = 2$, . . . $\cos. \frac{2 \times 2\pi + \phi}{n} + \sqrt{-1} \sin. \frac{2 \times 2\pi + \phi}{n}$;

. . . . . .

$m = n - 1$. $\cos. \frac{(n-1)2\pi + \varphi}{n} + \sqrt{-1} \sin. \frac{(n-1)2\pi + \varphi}{n}$

These values are all different; for 0, 1, 2, ... $n - 1$, being each less than n, the arcs $\frac{0}{n}, \frac{2\pi}{n}, \frac{2 \times 2\pi}{n} \ldots \frac{(n-1)2\pi}{n}$

are different, and each is less than a circumference: the values of the expressions $\cos.\frac{\phi}{n}$, $\sin.\frac{\phi}{n}$; $\cos.\frac{2\pi+\phi}{n}$, $\sin.\frac{2\pi+\phi}{n}\ldots$ $\cos.\frac{(n-1)2\pi+\phi}{n}$, $\sin.\frac{(n-1)2\pi+\phi}{n}$ are consequently all different; and their number is n.

If to m are given continually increasing values $m=n$, $m=n+1$, &c., the preceding values are reproduced.

For if $m=n$, $\cos.\frac{m\times 2\pi+\phi}{n}=\cos.\frac{n\times 2\pi+\phi}{n}=\cos.\left(2\pi+\frac{\phi}{n}\right)=\cos.\frac{\phi}{n}$. (Art. 30. m.)

$$\text{if } m=n+1,\ \cos.\frac{(n+1)\,2\pi+\phi}{n}=\cos.\left(2\pi+\frac{2\pi+\phi}{n}\right)=\cos.\frac{2\pi+\phi}{n}$$

Similarly, $\sin.\frac{n\times 2\pi+\phi}{n}=\sin.\frac{\phi}{n}$; $\sin.\frac{(n+1)2\pi+\phi}{n}=\sin.\frac{2\pi+\phi}{n}$: &c. &c.

Consequently, the second member of formula 2. contains all the values of which the first is susceptible, if the arc is successively made equal to $\phi, 2\pi+\phi, 2\times 2\pi+\phi\ldots\ldots\ldots$ $(n-1)\,2\pi+\phi$.

b. The preceding observations are also applicable to formula 3., the n different values of which are obtained by successively replacing ϕ by $1\times 2\pi+\phi$, $2\times 2\pi+\phi,\ldots\ldots$ $(n-1)\,2\pi+\phi$, in the expression $\cos.\frac{m}{n}\phi+\sqrt{-1}\sin.\frac{m}{n}\phi$.

In using this formula, it is necessary to reduce the fractional expression $\frac{m}{n}$ to the lowest terms.

44. In the formula,

$$\cos.n\phi+\sqrt{-1}\sin.n\phi=(\cos.\phi+\sqrt{-1}\sin.\phi)^n, \quad \ldots\ldots \quad 1.$$

let n be integer and positive.

Then if the arc ϕ is changed to $-\phi$, $\cos.(-\phi)=\cos.\phi$, $\sin.(-\phi)=-\sin.\phi$; and the formula becomes

$$\cos.n\phi-\sqrt{-1}\sin.n\phi=(\cos.\phi-\sqrt{-1}\sin.\phi)^n \quad \ldots\ldots \quad 2.$$

Subtracting Eq. 2. from Eq. 1.,

$$\sin.n\phi=\frac{(\cos.\phi+\sqrt{-1}\sin.\phi)^n-(\cos.\phi-\sqrt{-1}\sin.\phi)^n}{2\sqrt{-1}} \quad \ldots\ldots \quad 3.$$

And adding Eq. 2. to Eq. 1.,

$$\cos.n\phi=\frac{(\cos.\phi+\sqrt{-1}\sin.\phi)^n+(\cos.\phi-\sqrt{-1}\sin.\phi)^n}{2} \quad \ldots\ldots \quad 4.$$

Forming the powers of $(\cos.\phi+\sqrt{-1}\sin.\phi)^n$, $(\cos.\phi-\sqrt{-1}\sin.\phi)^n$ by the Binomial theorem (Alg. Art. 139.); and omitting the terms which destroy each other, it is found that,

$$\sin. n\phi = \frac{n}{1}\cos.^{n-1}\phi\,\sin.\phi - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cos.^{n-3}\phi\,\sin.^3\phi + \frac{n(n-1)(n-2)(n-3\,(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\cos.^{n-5}\phi\,\sin.^5\phi -, \&c. \quad 5.$$

$$\cos. n\phi = \cos.^n\phi - \frac{n(n-1)}{1\,.\,2}\cos.^{n-2}\phi\,\sin.^2\phi + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\cos.^{n-4}\phi\,\sin.^4\phi -, \&c. \quad . \quad . \quad . \quad . \quad 6.$$

Formulæ 5. and 6. express the sine and cosine of the multiple arc $n\phi$, in powers of the sine and cosine of the arc ϕ. The law of the terms is evident: and they are continued until a term is found which is equal to zero.

45. To express the sine and cosine of any arc in terms of the arc.

Let n be a whole number, so assumed that $n\phi$ is equal to any arc x.

Then, since $n\phi = x$,

$$n = \frac{x}{\phi}$$

$$n(n-1) = \frac{x}{\phi}\left(\frac{x}{\phi} - 1\right) = \frac{x}{\phi}\left(\frac{x-\phi}{\phi}\right) = \frac{x(x-\phi)}{\phi^2}.$$

$$n(n-1)(n-2) = \frac{x}{\phi}\left(\frac{x}{\phi}-1\right)\left(\frac{x}{\phi}-2\right) = \frac{x}{\phi}\left(\frac{x-\phi}{\phi}\right)\left(\frac{x-2\phi}{\phi}\right) = \frac{x(x-\phi)(x-2\phi)}{\phi^3}.$$

In the same manner,

$$n(n-1)(n-2)(n-3) = \frac{x(x-\phi)(x-2\phi)(x-3\phi)}{\phi^4},$$

$$n(n-1)(n-2)(n-3)(n-4) = \frac{x(x-\phi)(x-2\phi)(x-3\phi)(x-4\phi)}{\phi^5}, \&c.$$

Substituting these values for $n\phi$, n, $n(n-1)$, $n(n-1)(n-2)$, &c., in equations 5. and 6. of Art. 44.

$$\sin. x = \frac{x}{1.\phi}\cos.^{n-1}\phi\,\sin.\phi - \frac{x(x-\phi)(x-2\phi)}{1\,.\,2\,.\,3\,.\,\phi^3}\cos.^{n-3}\phi\,\sin.^3\phi + \frac{x(x-\phi)(x-2\phi)(x-3\phi)(x-4\phi)}{1\,.\,2\,.\,3\,.\,4\,.\,5\phi^5}\cos.^{n-5}\phi\,\sin.\phi^5 -, \&c.$$

$$\text{or } \sin. x = \frac{x}{1}\cos.^{n-1}\phi\,\frac{\sin.\phi}{\phi} - \frac{x(x-\phi)(x-2\phi)}{1\,.\,2\,.\,3}\cos.^{n-3}\phi\,\frac{\sin.^3\phi}{\phi^3} + \frac{x(x-\phi)(x-2\phi)(x-3\phi)(x-4\phi)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\cos.^{n-5}\phi\,\frac{\sin.^5\phi}{\phi^5} -, \&c.$$

$$\cos. x = \cos.^n\phi - \frac{x(x-\phi)}{1.2.\phi^2}\cos.^{n-2}\phi\,\sin.^2\phi + \frac{x(x-\phi)(x-2\phi)(x-3\phi)}{1\,.\,2\,.\,3\,.\,4\,.\,\phi^4}\cos.^{n-4}\phi\,\sin.^4\phi -, \&c.$$

$$\text{or } \cos. x = \cos.^n\phi - \frac{x(x-\phi)}{1\,.\,2}\cos.^{n-2}\phi\,\frac{\sin.^2\phi}{\phi^2} + \frac{x(x-\phi)(x-2\phi)(x-3\phi)}{1\,.\,2\,.\,3\,.\,4}\cos.^{n-4}\phi\,\frac{\sin.^4\phi}{\phi^4} -, \&c.$$

If now ϕ is indefinitely diminished so as scarcely to differ from 0, n must be indefinitely augmented so as scarcely to differ from ∞, in order that the arc $n\phi$ or x may have a finite magnitude. These are conditions which must be admitted in every investigation in which it is sought to express the sine and cosine of an arc in terms of that arc.

Now when ϕ is indefinitely diminished $\cos.\phi = 1$ and $\frac{\sin.\phi}{\phi} = 1$ (Art. 30. *n.*). Admitting that the powers of $\cos.\phi$ and $\frac{\sin.\phi}{\phi}$ are, under these conditions, also each equal to 1, ϕ and n as well as the powers of cos. ϕ and $\frac{\sin.\phi}{\phi}$ disappear from the formulæ, which are reduced to

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \&c.$$

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

a. If the series which expresses the value of sin. x is divided by the series which expresses the value of cos. x, the result is the expression of tan. x in terms of the arc x.

$$\therefore \tan. x = \frac{\sin. x}{\cos. x} = \frac{x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} -, \&c.}{1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} -, \&c.} = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3.3.5.7} +, \&c.$$

46. To express any arc in terms of the sine of that arc.

Let the arc be denoted by ϕ; also let $x = \sin. \phi$, and therefore $\sqrt{1 - x^2} = \cos. \phi$:

And assume the indeterminate series $\phi = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 +, \&c. \ldots\ldots 1.$

$$\phi' = \text{A} + \text{B}x' + \text{C}x'^2 + \text{D}x'^3 +, \&c. \ldots\ldots 2.$$

in which when $x = 0$, $\phi = 0$, and therefore $\text{A} = 0$.

Subtracting 2 from 1,

$$\phi - \phi' = \text{B}(x - x') + \text{C}(x^2 - x'^2) + \text{D}(x^3 - x'^3) + \text{E}(x^4 - x'^4) +, \&c.$$

Dividing both members of the result by $x - x'$ (Alg. Art. 22.)

$$\frac{\phi - \phi'}{x - x'} = \text{B} + \text{C}(x + x') + \text{D}(x^2 + xx' + x'^2) + \text{E}(x^3 + x^2x' + x'x^2 + x'^3) +, \&c. \ldots 3.$$

Now, $x - x' = \sin. \phi - \sin. \phi' = \left(\phi - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} -, \&c.\right) - \left(\phi' - \frac{\phi'^3}{1.2.3} + \frac{\phi'^5}{1.2.3.4.5} -, \&c.\right.$ (Art. 45.)

$$= \phi - \phi' - \frac{\phi^3 - \phi'^3}{1.2.3} + \frac{\phi^5 - \phi'^5}{1.2.3.4.5} - \&c.$$

$$\therefore \frac{\phi - \phi'}{x - x'} = \frac{\phi - \phi'}{\phi - \phi' - \frac{\phi^3 - \phi'^3}{1.2.3} + \frac{\phi^5 - \phi'^5}{1.2.3.4.5} - \&c.}$$

$$= \frac{1}{1 - \frac{1}{1.2.3}(\phi^2 + \phi\phi' + \phi'^2) + \frac{1}{1.2.3.4.5}(\phi^4 + \phi^3\phi' + \phi^2\phi'^2 + \phi\phi'^3 + \phi'^4), \&c.}$$

Substituting this value of $\frac{\phi - \phi'}{x - x'}$ in equation 3,

$$\frac{1}{1 - \frac{1}{1.2.3}(\phi^2 + \phi\phi' + \phi'^2) +, \&c.} = \text{B} + \text{C}(x + x') + \text{D}(x^2 + xx' + x'^2) +, \&c. \ldots 4.$$

Making $\phi' = \phi$, and consequently $x' = x$, equation 4 becomes

$$\frac{1}{1-\frac{\phi^2}{1.2}+\frac{\phi^4}{1.2.3.4}}=\text{B}+2\,\text{C}x+3\,\text{D}x^2+4\,\text{E}x^3+\&\text{c.}$$

$$\text{But } \frac{1}{1-\frac{\phi^2}{1.2}+\frac{\phi^4}{1.2.3.4}}=\frac{1}{\cos.\phi}\ (\text{Art. 45.})=\frac{1}{\sqrt{1-\sin.^2\phi}}=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}.$$

$$\text{And } (1-x^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\&\text{c.}\quad(\text{Alg. Art. 175.})$$

$$\therefore 1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\&\text{c.}=\text{B}+2\text{C}x+3\text{D}x^2+4\text{E}x^3+\&\text{c.}$$

Since this equation is true for every value of x, the coefficients of the same powers of x, in the two members must be equal. (Alg. Art. 174.)

$$\begin{array}{rlcrl}
\text{Hence }\ \text{B}= & 1 & \therefore & \text{B}= & 1\\
2\,\text{C}= & 0\ . & . & \text{C}= & 0\\
3\,\text{D}= & \tfrac{1}{2}\ . & . & \text{D}= & \frac{1}{2.3}\\
4\,\text{E}= & 0\ . & . & \text{E}= & 0\\
5\,\text{F}= & \frac{1.3}{2.4}\ . & . & \text{F}= & \frac{1.3}{2.4.5}\\
6\,\text{G}= & 0\ . & . & \text{G}= & 0\\
7\,\text{H}= & \frac{1.3.5}{2.4.6}\ . & . & \text{H}= & \frac{1.3.5}{2.4.6.7}
\end{array}$$

It has been already found that $\text{A}=0$. Substituting these values of A, B, C, &c. in the series

$$\phi=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+\text{E}x^4+\text{F}x^5+\text{G}x^6+\text{H}x^7+\&\text{c.}$$

$$\phi=x+\frac{1}{2.3}x^3+\frac{1.3}{2.4.5}x^5+\frac{1.3.5}{2.4.6.7}x^7+\&\text{c.}$$

$$\text{or } \phi=\sin.\phi+\frac{1}{2.3}\sin.^3\phi+\frac{3}{2.4.5}\sin.^5\phi+\frac{3.5}{2.4.6.7}\sin.^7\phi+\&\text{c.}$$

a. To express arc ϕ in terms of the powers of cos. ϕ.

Let $x=\cos.\phi$; $\sqrt{1-x^2}=\sin.\phi$; and assume the indeterminate series

$$\phi=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+\&\text{c.};\ \phi'=\text{A}+\text{B}x'+\text{C}x'^2+\text{D}x'^3+\&\text{c.}$$

in which, when $x=0$, or $\cos.\phi=0$, arc $\phi=\frac{\pi}{2}$, and therefore $\text{A}=\frac{\pi}{2}$.

If the series ϕ' is subtracted from the series ϕ, and both members of the result are divided by $x-x'$,

$$\frac{\phi-\phi'}{x-x'}=\text{B}+\text{C}\,(x+x')+\text{D}\,(x^2+xx'+x'^2)+\&\text{c.}$$

And if for $x-x'$ is substituted the value of the equivalent expression $\cos.\phi-\cos.\phi'$, which is

$$-\tfrac{1}{2}\,(\phi^2-\phi'^2)+\frac{1}{1.2.3.4}(\phi^4-\phi'^4)-\&\text{c.}\quad(\text{Art. 45.}),$$

and the terms of the result are divided by $\phi - \phi'$,

$$\frac{1}{-\frac{1}{2}(\phi+\phi')+\frac{1}{1.2.3.4}(\phi^3+\phi^2\phi'+\phi\phi'^2+\phi'^3)-\&c.}=B+C(x+x')+D(x^2+xx'+x'^2)+\&c.$$

Making $\phi=\phi'$ and therefore $x=x'$, this expression becomes

$$\frac{1}{-\phi+\frac{\phi^3}{1.2.3}-\frac{\phi^5}{1.2.3.4.5}+\&c.}=B+2Cx+3Dx^2+4Ex^3+\&c.$$

But

$$\frac{1}{-\phi+\frac{\phi^3}{1.2.3}-\&c.}=\frac{1}{-\left(\phi-\frac{\phi^3}{1.2.3}+\&c.\right.}=\frac{1}{-\sin.\phi}\text{ (Art. 45.)}=\frac{1}{-\sqrt{1-x^2}}=-(1-x^2)^{-\frac{1}{2}}$$

And $-(1-x^2)^{-\frac{1}{2}}=-1-\frac{1}{2}x^2-\frac{1.3}{2.4}x^4-\frac{1.3.5}{2.4.6}x^6-\&c.$ (Alg. Art. 175.)

$$\therefore\ -1-\tfrac{1}{2}x^2-\frac{1.3}{2.4}x^4-\frac{1.3.5}{2.4.6}x^6-\&c.=B+2Cx+3Dx^2+4Ex^3+\&c.$$

Consequently, as the co-efficients of the same powers of x are equal (Alg. Art. 174.)

$B=-1$; $C=0$; $D=-\frac{1}{2.3}$; $E=0$; $F=-\frac{1.3}{2.4.5}$, &c. Also, $A=\frac{\pi}{2}$.

If these values are substituted for A, B, C, &c. in the series $\phi=A+Bx+Cx^2+\&c.$

$$\phi=\frac{\pi}{2}-x-\frac{1}{2.3}x^3-\frac{1.3}{2.4.5}x^5-\frac{1.3.5}{2.4.6.7}x^7-\&c.$$

or $\phi=\frac{\pi}{2}-\cos.\phi-\frac{1}{2.3}\cos.^3\phi-\frac{1.3}{2.4.5}\cos.^5\phi-\frac{1.3.5}{2.4.6.7}\cos.^7\phi-\&c.$

b. The arc whose sine is x is frequently denoted by the symbol $\sin.^{-1}x$ and in like manner the arc whose cosine is x, by $\cos.^{-1}x$; the arc whose tangent is x by $\tan.^{-1}x$, &c.

Employing this notation, the formulæ of this article are expressed thus:

$$\sin.^{-1}x=x+\frac{1}{2.3}x^3+\frac{1.3}{2.4.5}x^5+\frac{1.3.5}{2.4.6.7}x^7+\&c.;\ \ldots\ 1.$$

$$\cos.^{-1}x=\frac{\pi}{2}-x-\frac{1}{2.3}x^3-\frac{1.3}{2.4.5}x^5-\frac{1.3.5}{2.4.6.7}x^7-\&c.\ldots 2.$$

x, x^3, &c. denoting sin., sin.^{3}, &c. of the arc in formula 1, and cos., cos.^{3}, &c. of the arc in formula 2.

In the second members, x is expressed in terms of radius; and consequently the first member will be the value of an arc whose radius $=1$.

If the first members be required in seconds, it is manifest that the sums of the terms in the second members must be divided by arc 1″ or sin. 1″.

47. If in the expressions

$$\left.\begin{array}{l} \varepsilon^{x}=1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\text{\&c.} \\ \varepsilon^{-x}=1-\frac{x}{1}+\frac{x^2}{2}-\frac{x^3}{2.3}+\frac{x^4}{2.3.4}-\text{\&c.} \end{array}\right\}\text{(Appen. Art. 10.)}$$

x is replaced by $x\sqrt{-1}$ and $-x$ by $-x\sqrt{-1}$, the results are

$$\varepsilon^{x\sqrt{-1}}=1+\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}-\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5\sqrt{-1}}{1.2.3.4.5}-\text{\&c.}$$

$$\varepsilon^{-x\sqrt{-1}}=1-\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}+\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}-\frac{x^5\sqrt{-1}}{1.2.3.4.5}-\text{\&c.}$$

Adding the second of these expressions to the first; then subtracting the second from the first, and dividing the results by 2 and $2\sqrt{-1}$ respectively,

$$\frac{\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}}{2}=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{\&c.}\;\ldots\ldots\;1$$

$$\frac{\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}}{2\sqrt{-1}}=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\text{\&c.}\;\ldots\ldots\;2$$

The second members of formulæ 1, 2, express values of cos. x and sin. x respectively (Art. 45.):

$$\therefore \cos. x=\frac{\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}}{2}=\tfrac{1}{2}\left(\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}\right)\;\ldots\ldots\;3$$

$$\sin. x=\frac{\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}}{2\sqrt{-1}}=\frac{1}{2\sqrt{-1}}\left(\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}\right).\;\ldots\;4$$

Dividing Eq. 4. by Eq. 3., and substituting tan. x for $\frac{\sin. x}{\cos. x}$,

$$\tan. x=\frac{1}{\sqrt{-1}}\left(\frac{\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}}{\varepsilon_{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}}\right).\;\ldots\ldots\;5$$

or, multiplying both terms of the second member by $\varepsilon^{x\sqrt{-1}}$,

$$\tan. x=\frac{1}{\sqrt{-1}}\left(\frac{\varepsilon^{2\sqrt{-1}}-1}{\varepsilon^{2x\sqrt{-1}}+1}\right).\;\ldots\ldots\;6$$

a. If nx is substituted for x in formulæ 3, 4 and 5,

$$\cos. nx=\tfrac{1}{2}\left(\varepsilon^{nx\sqrt{-1}}+\varepsilon^{-nx\sqrt{-1}}\right)$$

$$\sin. nx=\frac{1}{2\sqrt{-1}}\left(\varepsilon^{nx\sqrt{-1}}-\varepsilon^{-nx\sqrt{-1}}\right)$$

$$\tan. nx=\frac{1}{\sqrt{-1}}\left(\frac{\varepsilon^{nx\sqrt{-1}}-\varepsilon^{-nx\sqrt{-1}}}{\varepsilon^{nx\sqrt{-1}}+\varepsilon^{-nx\sqrt{-1}}}\right)=\frac{1}{\sqrt{-1}}\,\frac{\varepsilon^{2nx\sqrt{-1}}-1}{\varepsilon^{2nx\sqrt{-1}}+1}.$$

b. If both members of Eq. 5. are multiplied by $\sqrt{-1}$,

$$\sqrt{-1}\ \text{tan.}\ x = \sqrt{-1}\ \frac{\text{sin.}\ x}{\text{cos.}\ x} = \frac{\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}}{\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}}.$$

c. Let $\varepsilon^{x\sqrt{-1}} = \theta,\ \varepsilon^{-x\sqrt{-1}} = \frac{1}{\theta}$:

Then, $2\ \text{cos.}\ x = \theta + \frac{1}{\theta}$ (Art. 47. Form. 3.); and $2\sqrt{-1}\ \text{sin.}\ x = \theta - \frac{1}{\theta}$. (Art. 47. Form. 4.)

Also, $2^n\ \text{cos.}^n x = \left(\theta + \frac{1}{\theta}\right)^n = \left(\frac{1}{\theta} + \theta\right)^n$.

$$= \theta^n + n\theta^{n-2} + \frac{n(n-1)}{1\ .\ 2}\ \theta^{n-4} + \&\text{c.} \quad .\ .\ .\ .\ .\ 1.$$

$$= \frac{1}{\theta^n} + n\frac{1}{\theta^{n-2}} + \frac{n(n-1)}{1\ .\ 2}\ \frac{1}{n-4} + \&\text{c.} \quad .\ .\ .\ .\ .\ 2.$$

Adding eq. 1, 2, together, and observing that $2(2^n\ \text{cos.}^n x) = 2^{n+1} \text{cos.}^n x$;

$$2^{n+1}\ \text{cos.}^n x = \theta^n + \frac{1}{\theta^n} + n\left(\theta^{n-2} + \frac{1}{\theta^{n-2}}\right) + \frac{n(n-1)}{1\ .\ 2}\left(\theta^{n-4} + \frac{1}{\theta^{n-4}}\right) + \&\text{c.}$$

But $\theta^n + \frac{1}{\theta^n}$, whatever be $n = 2\ \text{cos.}\ nx$.

$$\therefore 2^n\ \text{cos.}^n x = \text{cos.}\ nx + n\ \text{cos.}\ (n-2)x + \frac{n(n-1)}{1\ .\ 2}\ \text{cos.}\ (n-4)x + \&\text{c.}$$

d. $2\sqrt{-1}\ \text{sin.}\ x = \theta - \frac{1}{\theta}$ $\therefore 2^n (\sqrt{-1})^n \text{sin.}^n x = \left(\theta - \frac{1}{\theta}\right)^n$, or $= \pm\left(\frac{1}{\theta} - \theta\right)^n$, the positive sign being used when n is even; the negative sign, when n is odd.

When n is even, $(\sqrt{-1})^n$ disappears, and

$$2^n\ \text{sin.}^n x = \theta^n - n\theta^{n-2} + \frac{n(n-1)}{1\ .\ 2}\ \theta^{n-4} - \&\text{c.}$$

$$= \frac{1}{\theta^n} - n\ \frac{1}{\theta^{n-2}} + \frac{n(n-1)}{1\ .\ 2}\ \frac{1}{\theta^{n-4}} - \&\text{c.}$$

Adding, and proceeding as before;

$$2^n\ \text{sin.}^n x = \text{cos.}\ nx - n\ \text{cos.}\ (n-2)x + \frac{n(n-1)}{1\ .\ 2}\ \text{cos.}\ (n-4)x - \&\text{c.}$$

When n is odd, $2^n(\sqrt{-1})^n\ \text{sin.}^n x = \theta^n - n\theta^{n-2} + \frac{n(n-1)}{1\ .\ 2}\ \theta^{n-4} - \&\text{c.}$

$$= -\frac{1}{\theta^n} + n\frac{1}{\theta^{n-2}} - \frac{n(n-1)}{2}\frac{1}{\theta^{n-4}} + \&\text{c.}$$

Adding the series together,

$$2^{n+1}(\sqrt{-1})^n\ \text{sin.}^n x = \theta^n - \frac{1}{\theta^n} - n\left(\theta^{n-2} - \frac{1}{\theta^{n-2}}\right) + \&\text{c.}$$

$$= 2\sqrt{-1}\ \text{sin.}\ nx - 2n\sqrt{-1}\ \text{sin.}\ (n-2)x + \&\text{c.}$$

or, $2^n(\sqrt{-1})^{n-1}\ \text{sin.}^n x = \text{sin.}\ nx - n\ \text{sin.}\ (n-2)x + \&\text{c.}$

in which $(\sqrt{-1})^{n-1}$ disappears because $n-1$ is even.

The co-efficients of the cosines and sines are identical with those of the binomial theorem, and those which are at equal distances from the extreme terms are equal to one another.

When n is even, one of the co-efficients of x is 0; and in this case twice the sum of the terms from $\cos. nx$ to $\cos. 0x$, is the sum of the whole series or the value of $2^n \cos.^n x$.

When n is odd, the term containing $\cos. 0x$ is the middle term: in which case, twice the sum of the preceding terms together with the middle term, is the sum of the series $2^n \cos.^n x$.

e. Since $\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}} = 2\cos. x$ (Formula 3.)

And $\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}} = 2\sqrt{-1} \sin. x$ (Formula 4.)

$\therefore$ by addition,

$$2\varepsilon^{x\sqrt{-1}} = 2\cos. x + 2\sqrt{-1} \sin. x, \text{ and } \varepsilon^{x\sqrt{-1}} = \cos. x + \sqrt{-1} \sin. x.$$

And by subtraction,

$$2\varepsilon^{-x\sqrt{-1}} = 2\cos. x - 2\sqrt{-1} \sin. x, \text{ and } \varepsilon^{-x\sqrt{-1}} = \cos. x - \sqrt{-1} \sin. x$$

$$\therefore \frac{\varepsilon^{x\sqrt{-1}}}{\varepsilon^{-x\sqrt{-1}}} \text{ or } \varepsilon^{2x\sqrt{-1}} = \frac{\cos. x + \sqrt{-1} \sin. x}{\cos. x - \sqrt{-1} \sin. x},$$

or, dividing all the terms of the second member of this equation by $\cos. x$,

$$\varepsilon^{2x\sqrt{-1}} = \frac{\dfrac{\cos. x}{\cos. x} + \sqrt{-1}\,\dfrac{\sin. x}{\cos. x}}{\dfrac{\cos. x}{\cos. x} - \sqrt{-1}\,\dfrac{\sin. x}{\cos. x}} = \frac{1 + \sqrt{-1} \tan. x}{1 - \sqrt{-1} \tan. x}.$$

Taking the logarithms of both members, in Napier's system (Appendix, Art. 10.),

$$l_{\varepsilon}\left(\varepsilon^{2x\sqrt{-1}}\right) = l_{\varepsilon}\left(\frac{1 + \sqrt{-1} \tan. x}{1 - \sqrt{-1} \tan. x}\right).$$

But the logarithm of the first member, in Napier's system, is $2x\sqrt{-1}$ (App. Art. 10.); and the log. of the second is

$$2\left(\frac{\sqrt{-1} \tan. x}{1} + \frac{(\sqrt{-1} \tan. x)^3}{3} + \frac{(\sqrt{-1} \tan. x)^5}{5} + \&c\right)$$

$$\therefore 2x\sqrt{-1} = 2\left\{\frac{\sqrt{-1} \tan. x}{1} + \frac{(\sqrt{-1} \tan. x)^3}{3} + \frac{(\sqrt{-1} \tan. x)^5}{5} + \&c.\right\}$$

Dividing every term of this expression by $2\sqrt{-1}$,

$$x = \frac{\tan. x}{1} + (\sqrt{-1})^2\,\frac{\tan.^3 x}{3} + (\sqrt{-1})^4\,\frac{\tan.^5 x}{5} + \&c.$$

Now, $(\sqrt{-1})^2 = -1$; $(\sqrt{-1})^4 = (-1) \times (-1) = +1$; $(\sqrt{-1})^6 = (-1) \times (-1) \times (-1) = -1$; &c.

$$\therefore x = \tan. x - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \frac{\tan.^7 x}{7} + \&c.$$

This is Gregory's formula. It may be employed in the calculation of the value of $\tan.^{-1} x$, when $\tan. x$ is known.

Let x be an arc of 30°:

Then $x=\frac{180^\circ}{6}=\frac{\pi}{6}$: tan. $x=$ tan. $30^\circ=\frac{1}{\sqrt{3}}$ (Art. 33. *c.*)

$$\text{And } \frac{\pi}{6}=\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3^3}}+\frac{1}{5\sqrt{3^5}}-\frac{1}{7\sqrt{3^7}}+\text{\&c.}$$

$$\text{Or } \frac{\pi}{6}=\frac{1}{\sqrt{3}}-\frac{1}{3.3\sqrt{3}}+\frac{1}{5.3^2\sqrt{3}}-\frac{1}{7.3^3\sqrt{3}}+\text{\&c.}$$

$$\therefore \pi=\frac{2\sqrt{3}}{1}-\frac{2\sqrt{3}}{3.3}+\frac{2\sqrt{3}}{5.3^2}-\frac{2\sqrt{3}}{7.3^3}+\text{\&c.}$$

On account of the radical quality, $\sqrt{3}$, the labour of determining π from this series is very considerable.

If x is an arc of 45°,

$$x=\frac{180^\circ}{4},=\frac{\pi}{4} \text{ and tan. } x=\text{tan. } 45^\circ=1 \text{ (Art. 33. } d.)$$

$$\therefore \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{\&c.}$$

This series converging too slowly to be of any practical use; suppose that the arc 45° is divided into two arcs A and B:

Then, since $\tan.(\text{A}+\text{B})=\frac{\tan.\text{A}+\tan.\text{B}}{1-\tan.\text{A}\tan.\text{B}}$ (Art. 37.), and $\tan.(\text{A}+\text{B})=\tan.45^\circ=1$.

$$1=\frac{\tan.\text{ A}+\tan.\text{ B}}{1-\tan.\text{ A}.\tan.\text{ B}},$$

$$\therefore 1-\tan.\text{ A}.\tan.\text{ B}=\tan.\text{ A}+\tan.\text{ B}.$$

$$\text{and } \tan.\text{ B}=\frac{1-\tan.\text{ A}}{1+\tan.\text{ A}}.$$

Giving to tan. A the arbitrary value $\frac{1}{2}$,

$$\tan.\text{ B}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{2}\div\frac{3}{2}=\frac{1}{3}.$$

Consequently the length of the arc A is given by the series,

$$\text{A}=\frac{1}{1.2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}+\frac{1}{9.2^9}-\text{\&c.}$$

and that of the arc B by the series

$$\text{B}=\frac{1}{1.3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\frac{1}{9.3^9}-\text{\&c.}$$

Whence, since the sum of the arcs A, B is $45^\circ=\frac{\pi}{4}$,

$$\pi=4\left\{\begin{array}{l}\frac{1}{1.2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}+\frac{1}{9.2^9}-\text{\&c.}\\ +\left(\frac{1}{1.3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\frac{1}{9.3^9}-\text{\&c.}\right)\end{array}\right\}.$$

The value of π can be found more easily from this than from the preceding series, which involves the radical quantity $\sqrt{3}$.

48. The numerical values of the sines and cosines of 15°, 30°, 45°, and 75°, which are given in Art. 33. may be employed to test the accuracy of the trigonometrical tables. The values of the cosines of 36° and 72° are useful for the same purpose.

Now, from the definition of the sine of an arc (Art. 30. *a.*), and from Euc. III. 3. it follows that the sine of any arc is equal to half the chord of twice that arc. Therefore the sine of an arc of 18° is equal to half the chord of an arc of 36°. But an arc of 36° is the tenth part of the whole circumference: and therefore the chord of 36° is equal to the side of the inscribed decagon. Again, it may be inferred from the construction of the problems Euc. IV. 10. and VI. 30., that the side of a decagon inscribed in any circle is equal to the greater segment of the radius of that circle, divided in extreme and mean ratio.

Hence, making rad. $=1$, and side of decagon $=x$,

$$x^2=1\,(1-x)=1-x$$

Consequently (Alg. Art. 119.), $x=\frac{1}{2}(\sqrt{5}-1)=$ chord of 36°,

$$\therefore\ \sin.\ 18^\circ=\frac{1}{4}(\sqrt{5}-1)$$

$$\therefore\ \cos.\ 72^\circ=\frac{1}{4}(\sqrt{5}-1).$$

Next, to find $\cos.\ 36^\circ=\cos.\ 2\times 18^\circ=\cos.^2 18^\circ-\sin.^2 18^\circ$. (Art. 34. *a.*)

$$\cos.^2 18^\circ=1-\sin.^2 18^\circ=1-(\tfrac{1}{4}(\sqrt{5}-1))^2=1-\tfrac{1}{16}(5-2\sqrt{5}+1)=\tfrac{1}{16}(10+2\sqrt{5})$$

$$\therefore\ \cos.\ 18^\circ=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

And $\cos.36^\circ=\tfrac{1}{16}(10+2\sqrt{5})-\tfrac{1}{16}(5-2\sqrt{5}+1)=\tfrac{4}{16}+\tfrac{4}{16}\sqrt{5}=\tfrac{1}{4}(\sqrt{5}+1)$.

Whence also $\sin.\ (90^\circ-36^\circ)$ or $\sin.\ 54^\circ=\frac{1}{4}(\sqrt{5}+1)$

a. In this section are investigated a considerable number of the more important trigonometrical formulæ. The variety of trigonometrical expressions is, however, endless; and unknown formulæ must, in consequence, be often met with. The easiest method of testing the accuracy of such is to verify them by means of other and known relations of the quantities of which they are composed.

Thus, to verify the expression

$$\sin.\,(A+B)\,\sin.\,(A-B)=\sin.^2 A-\sin.^2 B\,;$$

if $\sin.\,(A+B)$ and $\sin.\,(A-B)$ are replaced by their known values (Art. 31.),

$$\sin.\,(A+B)\,\sin.\,(A-B)=(\sin.A\cos.B+\cos.A\sin.B)(\sin.A\cos.B-\cos.A\sin.B),$$
$$=\sin.^2 A\cos.^2 B-\cos.^2 A\sin.^2 B.$$

Next, if $1-\sin.^2 B$, $1-\sin.^2 A$, are substituted for $\cos.^2 B$, $\cos.^2 A$,

$$\sin.(A+B)\sin.(A-B)=\sin.^2 A(1-\sin.^2 B)-\sin.^2 B(1-\sin.^2 A)=\sin.^2 A-\sin.^2 B.$$

The expression proposed is therefore reduced to the identity $\sin.^2 A-\sin.^2 B=\sin.^2 A-\sin.^2 B$.

To verify the expression $\cos.\,A=\dfrac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}$;

If for $\tan.^2\frac{1}{2}A$, is substituted its known value $\dfrac{\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}$, the numerator becomes $\dfrac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}$, and the denominator $\dfrac{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}$.

$$\therefore\ \frac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}=\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}\div\frac{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}=\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}.$$

But $\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A=\cos.\,A$ (Art. 34. *a.*); and $\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A=1$. (Art. 19. *d.*)

$\therefore\ \dfrac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}=\dfrac{\cos.\,A}{1}$, and the expression $\cos.\,A=\dfrac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}$ is reduced to the identity $\cos.\,A=\cos.\,A$.

SECTION IV.

OF THE CALCULATION OF THE TRIGONOMETRICAL TABLES.

49. In the sixth proposition of the tract on the circle (Euc. page 176.), it is proved that the perimeter of a regular polygon of a very great number of sides, circumscribed about or inscribed in a circle does not sensibly differ from the circumference of that circle. Now, by assuming a value of π, which is correct to 25 places of decimals, it has been found in Art. 30. *p.*, that the length of the arc of $1''$ (the radius being 1) is ·000004848136811107992+, and from this result (assuming that the arc of $2''$ does not differ from its chord), that the same number expresses the value of the sine of $1''$.

The accuracy of this determination can be tested by the formula of Art. 45.

$$\sin.\ \varphi = \varphi - \frac{\varphi^3}{2.3} + \frac{\varphi^5}{2.3.4.5} - \frac{\varphi^7}{2.3.4.5.6.7} + \&c.$$

For, putting the second member of this series under the form,

$$\varphi\left(1 - \frac{\varphi^2}{6} + \frac{\varphi^4}{120} - \frac{\varphi^6}{5040} + \&c.\right)$$

And considering the part $1 - \frac{\varphi^2}{6} + \frac{\varphi^4}{120} - \frac{\varphi^6}{5040} + \&c.$: if $\varphi = 0$, this expression is reduced to 1; and if φ, without being 0, is a very small arc, for example $1''$, the first term 1, which expresses the numerical value of an arc of $57^\circ.29577+$ (Art. 14. *f.*) must exceed the value of all the remaining terms of the series. Hence it is evident that, when φ is a very small arc, the value of each term of the series $\varphi - \frac{\varphi^3}{6} + \frac{\varphi^5}{120} - \&c.$ is greater than the sum of all the succeeding terms.

Consequently, the signs of the terms being alternately + and −, the value of sin. φ is less than that of the sum of any odd number of terms of the series, and greater than that of the sum of any even number of terms of the series.

Denoting therefore by φ an arc of $1''$,

$$\text{sin. } \varphi < \varphi,$$

$$\text{sin. } \varphi > \varphi - \frac{\varphi^3}{6}.$$

Now $\varphi = .00000484813681107992783713371+$

and $\frac{\varphi^3}{6} = .00000000000000001899211577633+$

$\therefore\ \varphi - \frac{\varphi^3}{6} = .00000484813681106093572135738\pm$,

The value of sin. $1''$ is therefore comprehended between two numbers which have the first sixteen figures common, and differ by less than two units of the seventeenth figure. These sixteen figures consequently belong to the true value of sin. $1''$.

The first significant figure of the correction $+\frac{\varphi^5}{120}$, falling in the twenty-ninth place, affects only the last figure of the last preceding approximate value of sin. $1''$.

The value of sin. $1''$ being determined, that of cos. $1''$ is obtained by substituting the numerical value of $\text{sin.}^2\, 1''$ in the formula $\text{cos. } \varphi = \sqrt{1 - \text{sin.}^2\, \varphi}$.

Making this substitution,

$\text{cos. } 1'' = \sqrt{1 - .0000000000235044305387550261383236+}$

or $\text{cos. } 1'' = \sqrt{.9999999999764955694612449738616764}$

or $\text{cos. } 1'' = .999999999988247 7847+$

Sin. $1''$ and cos. $1''$ being determined, sin. $2''$ and cos. $2''$ can be found by substituting the numerical values of sin. $1''$ and cos. $1''$ in the second members of the formulæ

$\text{sin. } 2\text{A} = 2 \text{ sin. A cos. A}$; $\text{cos. } 2\text{A} = \text{cos.}^2\, \text{A} - \text{sin.}^2\, \text{A}$, which thus become

$\text{sin. } 2'' = 2 \text{ sin. } 1'' \text{ cos. } 1''$; $\text{cos. } 2'' = \text{cos.}^2\, 1'' - \text{sin.}^2\, 1''$.

Or, more generally, the numerical values of sin. $2''$, cos. $2''$, sin. $3''$, cos. $3''$, sin. $4''$, cos. $4''$, &c., can be found from those of sin. $1''$, cos. $1''$ and from each other, by means of the formulæ

$$\text{sin. } (\text{A} + \text{B}) = \text{sin. A cos. B} + \text{cos. A sin. B}:$$
$$\text{cos. } (\text{A} + \text{B}) = \text{cos. A cos. B} - \text{sin. A sin. B}.$$

From these formulæ are thus successively obtained,

$$\begin{aligned}
&\sin.(1''+1'')=\sin.1''\cos.1''+\cos.1''\sin.1''\\
&\quad\text{or }\sin.2''=2\sin.1''\cos.1''\\
&\cos.(1''+1'')=\cos.1''\cos.1''-\sin.1''\sin.1''\\
&\quad\text{or }\cos.2''=\cos.^2 1''-\sin.^2 1''\\
&\sin.(2''+1'')=\sin.2''\cos.1''+\cos.2''\sin.1''\\
&\quad\text{or }\sin.3''=\sin.2''\cos.1''+\cos.2''\sin.1''\\
&\qquad\cos.3''=\cos.2''\cos.1''-\sin.2''\sin.1''\\
&\qquad\sin.4''=\sin.3''\cos.1''+\cos.3''\sin.1''\\
&\qquad\cos.4''=\cos.3''\cos.1''-\sin.3''\sin.1''\\
&\qquad\quad\&c.\qquad\&c.\qquad\&c.
\end{aligned}$$

a. The calculation of the numerical values of the sines and cosines of angles is made with less labour by means of two formulæ, which are found thus,

$$\begin{aligned}
\text{to }\sin.(A+B)&=\sin.A\cos.B+\cos.A\sin.B\\
\text{add }\sin.(A-B)&=\sin.A\cos.B-\cos.A\sin.B
\end{aligned}$$

$$\therefore\ \sin.(A+B)+\sin.(A-B)=2\sin.A\cos.B=2\cos.B\sin.A,$$

Transposing, $\sin.(A+B)=2\cos.B\sin.A-\sin.(A-B)$ . . . 1

In like manner,

from the formula $\cos.(A\pm B)=\cos.A\cos.B\mp\sin.A\sin.B$,
it is found that $\cos.(A+B)=2\cos.B\cos.A-\cos.(A-B)$ . . 2

The angles or arcs $A-B$, A, $A+B$, are three terms of an arithmetical progression, whose ratio is B: denoting $A-B$, A, $A+B$, by t, t', t'', respectively, formulæ 1, 2, become

$$\sin.t''=2\cos.B\sin.t'-\sin.t \quad . . . . 3$$
$$\cos.t''=2\cos.B\cos.t'-\cos.t \quad . . . . 4$$

Whence, if two consecutive terms, $\sin.t$, $\sin.t'$ or $\cos.t$, $\cos.t'$ are known, the third term $\sin.t''$ or $\cos.t''$ is obtained by multiplying the second term, $\sin.t'$ or $\cos.t'$ by $2\cos.B$ and subtracting the first term $\sin.t$, or $\cos.t$, from the product.

Consequently, making $B=1''$ and denoting the numerical values of $\sin.1''$, $\cos.1''$ by s and c, the sines and cosines, from second to second, are given by the following formulæ,

$$\begin{aligned}
\sin.0&=0\\
\sin.1''&=s\\
\sin.2''&=2c\sin.1''\\
\sin.3''&=2c\sin.2''-\sin.1''\\
\sin.4''&=2c\sin.3''-\sin.2''
\end{aligned}$$

. . . .

$$\begin{aligned}
\cos. 0 &= 1\\
\cos. 1'' &= c\\
\cos. 2'' &= 2c \cos. 1'' - 1\\
\cos. 3'' &= 2c \cos. 2'' - \cos. 1''\\
\cos. 4'' &= 2c \cos. 3'' - \cos. 2''.
\end{aligned}$$

b. The calculation of the numerical values of the sines and cosines from these formulæ is still susceptible of some abridgment.

For since $c = .9999999999882477+$

$1 - c = .0000000000117522+$

and $2 - 2c = .0000000000235044+$. Denoting this difference by d,

$$d = 2 - 2c, \text{ and } 2c = 2 - d.$$

Substituting $2 - d$ for $2c$ in formulæ 3. and 4,

$$\begin{aligned}
&\sin. t'' = (2 - d) \sin. t' - \sin. t\\
\text{or } &\sin. t'' = 2 \sin. t' - d \sin. t' - \sin. t\\
\text{or } &\sin. t'' - \sin. t' = (\sin. t' - \sin. t) - d \sin. t'. \quad . \quad . \quad . \quad 5
\end{aligned}$$

This formula shows that if the difference between the known terms sin. t' and sin. t is diminished by the product d sin. t', the result is equal to the difference sin. t'' — sin. t'. Now since sin. $t,'$ sin. t and d are known, this difference, sin. t'' — sin. t' can be found: then adding it to sin t', the result is sin. $t.''$

The only laborious process by this method is the calculation of d sin. t', an operation which must be repeated for every sine. The labour may, however, be abridged by forming at first a table of the products of the significant figures of d, viz. 235044, by the figures 1, 2, 3 . . . . . 9, and employing this table in the manner pointed out (Arith. Art. 61.).

The cosines may be calculated in the same manner from the analogous formula

$$\text{Cos. } t'' - \text{Cos. } t' = (\text{Cos. } t' - \text{Cos. } t) - d \text{ Cos. } t' \ldots\ldots 6.$$

If the numerical values of the sines and cosines are calculated, not from second to second, but from ten seconds to ten seconds, it will be necessary to substitute the numerical values of sin. 10″ and cos. 10″ for sin. 1″ and cos. 1″ in the preceding formulæ: and if they are calculated from minute to minute, the substitution of sin. 1′, cos. 1′ for sin 1″, cos. 1″ will, in like manner, be necessary.

c. When an angle exceeds 60°, the numerical value of its sine may be obtained by adding together the values of the sines of other angles which are each less than 60°. For, in the formula,

$$\begin{aligned}&\sin.(\text{B}+\text{A})-\sin.(\text{B}-\text{A})=2\cos.\text{B}.\sin.\text{A}\ (\text{Art. }39.),\\ &\text{or }\sin.(\text{B}+\text{A})=2\cos.\text{B}.\sin.\text{A}+\sin.(\text{B}-\text{A}),\\ &\text{if B}=60^\circ,\ \cos.\text{B}=\cos.60^\circ=\tfrac{1}{2}\ (\text{Art. }33.\ a).;\end{aligned}$$

$$\begin{aligned}&\therefore \sin.(60^\circ+\text{A})=2\times\tfrac{1}{2}\sin.\text{A}+\sin.(60^\circ-\text{A}),\\ &\text{or }\sin.(60^\circ+\text{A})=\sin.\text{A}.+\sin.(60^\circ-\text{A}).\end{aligned}$$

Hence to find, for example, the sine of 65° or sin (60° + 5°), it is only necessary to add the sine of the angle 5° to the sine of the angle (60° – 5°) or 55°, and the sum is the sine of the proposed angle.

The numerical values of the sines from 0 to 90° being computed, those of the cosines are given by the formula cos. A = sin. (90° – A).

d. If the numerical values of the sines and cosines of all angles from 0 to 30° are computed, the values of the sines and cosines of angles from 30° to 60° can be obtained from them by addition and subtraction. For, if in the formulæ,

$$\begin{aligned}&\sin.(\text{B}+\text{A})+\sin.(\text{B}-\text{A})=2\sin.\text{B}\cos.\text{A}\ (\text{Art. }39.),\\ &\cos.(\text{B}-\text{A})-\cos.(\text{B}+\text{A})=2\sin.\text{B}\sin.\text{A},\end{aligned}$$

B represents an angle of 30°, sin. B $=\frac{1}{2}$ (Art. 33. *b.*);
therefore 2 sin. B = 1,

$$\begin{aligned}\text{and consequently, }&\sin.(30^\circ+\text{A})+\sin.(30^\circ-\text{A})=\cos.\text{A};\\ &\cos.(30^\circ-\text{A})-\cos.(30+\text{A})=\sin.\text{A};\\ \text{or }&\sin.(30^\circ+\text{A})=\cos.\text{A}-\sin.(30^\circ-\text{A});\\ &\cos.(30^\circ+\text{A})=\cos.(30^\circ-\text{A})-\sin.\text{A}.\end{aligned}$$

$$\begin{aligned}\text{Making A}=1^\circ;\ &\sin.(30^\circ+1^\circ)=\cos.1^\circ-\sin.(30^\circ-1^\circ),\\ \text{or }&\sin.31^\circ=\cos.1^\circ-\sin.29^\circ\\ \text{Making A}=2^\circ\ &\sin.32^\circ=\cos.2^\circ-\sin.28^\circ\end{aligned}$$

. . . . .

$$\begin{aligned}\text{Similarly}\quad&\cos.31^\circ=\cos.29^\circ-\sin.1^\circ\\ &\cos.32^\circ=\cos.28^\circ-\sin.2^\circ\end{aligned}$$

. . . . .

It is not, however, necessary to continue the determination of the values of the sines and cosines, by this process, beyond sin. 45° and cos. 45°; for these being known, the values of the sines and cosines from 45° to 90°, are given by the formulæ sin. A = cos. (90° – A);
cos. A = sin. (90° – A);

50. The numerical values of the tangents and cotangents of angles from 0 to 45°, may be computed from the expressions

$$\text{tan. A} = \frac{\text{sin. A}}{\text{cos. A}} \text{ (Art. 19. } c\text{).}$$

$$\text{cot. A} = \frac{\text{cos. A}}{\text{sin. A}} \text{ (Art. 25. } b\text{).}$$

The tangents and cotangents of angles greater than 45° are determined from the cotangents and tangents of angles less than 45, by the formulæ

$$\text{tan. A} = \text{cot. } (90^\circ - \text{A});$$
$$\text{cot. A} = \text{tan. } (90^\circ - \text{A}).$$

Otherwise; the tangents of angles from 0 to 90° may be calculated by the formula $\text{tan. A} = \frac{\text{sin. A}}{\text{cos. A}}$; and the cotangents determined from the tangents by the formula $\text{cot. A} = \text{tan. } (90^\circ - \text{A})$. When this method is followed, it is only necessary to calculate the values of the tangents from 0 to 45°; after which the calculations may be abridged.

For since $\text{tan. } (\text{B} \pm \text{A}) = \frac{\text{tan. B} \pm \text{tan. A}}{1 \mp \text{tan. B tan. A}}$ (Art. 37.),

and since, if $\text{B} = 45^\circ$; $\text{tan. B} = 1$ (Art. 33. *d*),

$$\therefore \text{tan. } (45^\circ \pm \text{A}) = \frac{1 \pm \text{tan. A}}{1 \mp \text{tan. A}}$$

$$\therefore \text{tan. } (45^\circ + \text{A}) - \text{tan. } (45^\circ - \text{A}) = \frac{1 + \text{tan. A}}{1 - \text{tan. A}} - \frac{1 - \text{tan. A}}{1 + \text{tan. A}} = \frac{4 \text{ tan. A}}{1 - \text{tan.}^2 \text{A}} = 2 \frac{2 \text{ tan. A}}{1 - \text{tan.}^2 \text{A}}.$$

$$\text{But } 2 \times \frac{2 \text{ tan. A}}{1 - \text{tan.}^2 \text{A}} = 2 \times \text{tan. } 2\text{A} \text{ (Art. 38.),}$$

$$\therefore \text{tan. } (45^\circ + \text{A}) - \text{tan. } (45^\circ - \text{A}) = 2 \text{ tan. } 2\text{ A},$$
$$\text{or, tan. } (45^\circ + \text{A}) = 2 \text{ tan. } 2\text{A} + \text{tan. } (45^\circ - \text{A}).$$

Consequently the values of tan. A from 0 to 45° being known, the values from 45° to 90° are obtained from them by giving to A values from 0 to 45°, thus

$$\text{tan. } (45^\circ + 1^\circ) = 2 \text{ tan. } 2 \times 1^\circ + \text{tan. } (45^\circ - 1^\circ)$$
$$\text{or, tan. } 46^\circ = 2 \text{ tan. } 2^\circ + \text{tan. } 44^\circ:$$

$$\text{similarly, tan. } 47^\circ = 2 \text{ tan. } 4^\circ + \text{tan. } 43^\circ.$$

a. The numerical values of sec. A from 0 to 90° may be found from the formula

$$\text{sec. A} = \frac{1}{\text{cos. A}} \text{ (Art. 23. } b\text{.);}$$

and those of cosec. A from the formula

$$\text{cosec. A} = \text{sec. } (90^\circ - \text{A}).\ \ (\text{Art. 26. } b.)$$

The values of the secant and cosecant are not given in the trigonometrical tables.

51. It appears from Articles 49. and 50. that the values of all the trigonometrical ratios from 0 to 90°, proceeding from second to second, or in any other progression, may be derived from the values of sin. 1″, cos. 1″, and sin. 2″ by addition, merely. If this method is followed, an error in the value of the sine of any angle must of necessity affect all the results which follow it. It consequently becomes necessary to have some independent mode of testing the accuracy of the determinations. The numerical values of the sines and cosines of certain angles, determined in Articles 33. and 48., may be employed for this purpose. If the values of the sines, &c., of these angles agree with the values of the sines, &c., of the same angles, obtained by the processes of Articles 49. and 50., it may be concluded that the results are accurate.

The most general formula of verification is obtained from the expression

$\text{sin.}(\text{B} + \text{A}) - \text{sin.}(\text{B} - \text{A}) = 2 \text{ cos. B sin. A.}$ (Art. 39.) Thus,

1st. Let 36° be substituted for B, and the numerical value of cos. 36°, which (Art. 48.) is $\frac{1}{4}(\sqrt{5} + 1)$, for cos. B,

$$\therefore \text{sin.}(36^\circ + \text{A}) - \text{sin.}(36^\circ - \text{A}) = \tfrac{1}{2}(\sqrt{5} + 1) \text{ sin. A} \ \ldots\ldots\ 1$$

2d. Let next, 72° be substituted for B, and the numerical value of cos. 72°, which is $\frac{1}{4}(\sqrt{5} - 1)$ for cos. 72°;

$$\therefore \text{sin.}(72^\circ + \text{A}) - \text{sin.}(72^\circ - \text{A}) = \tfrac{1}{2}(\sqrt{5} - 1) \text{ sin. A} \ \ldots\ldots\ 2$$

Subtracting equation 2. from equation 1.,

$$\text{sin.}(36^\circ + \text{A}) - \text{sin.}(36^\circ - \text{A}) - \text{sin.}(72^\circ + \text{A}) + \text{sin.}(72^\circ - \text{A}) = \text{sin. A},$$
$$\text{or, sin. A} + \text{sin.}(36^\circ - \text{A}) + \text{sin.}(72^\circ + \text{A}) = \text{sin.}(36^\circ + \text{A}) + \text{sin.}(72^\circ - \text{A}).$$

This is the formula in question. In using it, different numbers expressing degrees, or minutes, or degrees and minutes, &c., are substituted for A; thus,

If $\text{A} = 1^\circ$, $\text{sin. } 1^\circ + \text{sin. } 35^\circ + \text{sin. } 73^\circ = \text{sin. } 37^\circ + \text{sin. } 71^\circ.$

Then adding the numerical values of sin. 1°, sin. 35°, and sin. 73° into one sum, and the numerical values of sin. 37° and sin. 71° into another, if the two results agree, it may be concluded that the sines of 1°, 35°, 73°, 37°, and 71° are accurately determined.

Similarly,
if A $= 2^\circ$; $\sin. 2^\circ + \sin. 34^\circ + \sin. 74^\circ = \sin. 38^\circ + \sin. 70^\circ$.

52. The values of the sines and cosines from 0 to 90° may also be found from the series

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \&c.,$$

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

by assuming $\pi = 3.14159 +$, making $x = \frac{m}{n}.\frac{\pi}{2}$, and assigning successive values to $\frac{m}{n}$, so that $\frac{m}{n}.\frac{\pi}{2}$ may be made to express, in succession, angles of $1''$, $2''$, $3'' \ldots 1'$, $2' \ldots 1^\circ$, $1^\circ\, 0'\, 1''$, &c.

If $\frac{m}{n}.\frac{\pi}{2}$ is substituted for x, the series for sin. x and cos. x become

$$\sin. \frac{m}{n}.\frac{\pi}{2} = \frac{m}{n}.\frac{\pi}{2} - \frac{m^3}{n^3}.\frac{\pi^3}{1.2.3.2^3} + \frac{m^5}{n^5}.\frac{\pi^5}{1.2.3.4.5.2^5} - \&c.$$

$$\cos. \frac{m}{n}.\frac{\pi}{2} = 1 - \frac{m^2}{n^2}.\frac{\pi^2}{1.2.2^2} + \frac{m^4}{n^4}.\frac{\pi^4}{1.2.3.4.2^4} - \&c.$$

By the substitution of numerical values for $\frac{\pi}{2}$, $\frac{\pi^2}{1.2.2^2}$, $\frac{\pi^3}{1.2.3.2^3}$, &c., these series are reduced to the following expressions: —

$$\begin{aligned} \text{Sin.} \frac{m}{n}.90^\circ = {} & 1.57079632679489619 \ldots \times \frac{m}{n} \\ & - .64596409750624625 \ldots \times \frac{m^3}{n^3} \\ & + .07969262624616705 \ldots \times \frac{m^5}{n^5} \\ & - .00468175413531869 \ldots \times \frac{m^7}{n^7} \\ & + .00016044118478736 \ldots \times \frac{m^9}{n^9} \\ & - .00000359884323521 \ldots \times \frac{m^{11}}{n^{11}} \\ & + .00000005692172922 \ldots \times \frac{m^{13}}{n^{13}} \\ & - .00000000066880351 \ldots \times \frac{m^{15}}{n^{15}} \\ & + .00000000000606694 \ldots \times \frac{m^{17}}{n^{17}} \\ & - .00000000000004377 \ldots \times \frac{m^{19}}{n^{19}} \\ & + .00000000000000026 \ldots \times \frac{m^{21}}{n^{21}} \\ & - \&c. \end{aligned}$$

$$\begin{aligned} \text{Cos.} \frac{m}{n}.90^\circ = {} & 1.00000000000000000 \ldots \times \frac{m^0}{n^0} \\ & - 1.23370055013616983 \ldots \times \frac{m^2}{n^2} \\ & + .25366950790104801 \ldots \times \frac{m^4}{n^4} \\ & - .02086348076335296 \ldots \times \frac{m^6}{n^6} \\ & + .00091926027483943 \ldots \times \frac{m^8}{n^8} \\ & - .00002520204237306 \ldots \times \frac{m^{10}}{n^{10}} \\ & + .00000047108747788 \ldots \times \frac{m^{12}}{n^{12}} \\ & - .00000000638660308 \ldots \times \frac{m^{14}}{n^{14}} \\ & + .00000000006565963 \ldots \times \frac{m^{16}}{n^{16}} \\ & - .00000000000052944 \ldots \times \frac{m^{18}}{n^{18}} \\ & + .00000000000000344 \ldots \times \frac{m^{20}}{n^{20}} \\ & - \&c. \end{aligned}$$

From these series are obtained the values of the sine and cosine of 1″ by making $\frac{m}{n} = \frac{1}{324000}$; the values of the sine and cosine of 1′ by making $\frac{m}{n} = \frac{1}{5400}$; the values of the sine and cosine of 1° by making $\frac{m}{n} = \frac{1}{90}$, &c.

Since it is sufficient, for the completion of the trigonometrical tables, to know the sines and cosines of angles from 0 to 30° (Art. 49. *d.*), the fraction $\frac{m}{n}$ need never be greater than $\frac{1}{3}$. These series must consequently converge very rapidly.

a. The values of the sines, cosines, &c., of arcs or angles, determined by the processes of Articles 49, 50. and 52. are called natural sines. The sines and cosines of angles from 1″ to 89° 59′ 59″, the tangents of angles from 1″ to 44° 59′ 59″, and the cotangents of angles from 45° 0′ 1″ to 89° 59′ 59″ (each inclusive), being less than 1, their values, if expressed decimally, contain no integer figures. In order that the values of all the ratios may be partly expressed by integer figures, the values which are given in the trigonometrical tables have been multiplied by 10000. The results, thus obtained, are said to be calculated to a radius of 10000.

53. In most of the calculations into which the trigonometrical ratios are made to enter, the numerical values of these ratios are not themselves employed, but their logarithms. The values of many of the ratios being less than 1 (Art. 52. *a.*) their logarithms must have negative characteristics. To obviate this inconvenience, the values of the sines, &c., are multiplied by $10000000000 = 10^{10}$, or the logarithms of their values are increased by 10. By this means the characteristics of the least numbers of the tables are rendered positive. For example, the natural sine of 1″ being .000004848 +, the characteristic of log. sin. 1″ is − 6. Increasing the characteristics by 10, that of log. sin. 1″ becomes + 4.

On account of the 10 added to the characteristic of the logarithm of every natural sine, cosine, &c. (or the multiplication of the value of the sine, cosine, &c., by 10^{10}), the logarithmic sines of the trigonometrical tables are said to be calculated to a radius of 10000000000.

The logarithms of the natural sines may be found from tables of the logarithms of numbers; or the logarithmic sines may be calculated immediately from the formulæ

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \&c.$$

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

The series which expresses sin. x may be put under the form

$$\sin. x = x\left(1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \&c.\right).$$

Then, if sin. $x = 0$, x may be equal to 0 or π, or 2π, or 3π . . . or $n\pi$, (n being any whole number): or x may be equal to $-\pi$ or -2π . . . . or $-n\pi$, &c.

Regarding the series

$$\sin. x (= 0) = x\left(1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \&c.\right)$$

as an equation, 0, π, 2π, 3π, &c, $-\pi$, -2π, -3π, &c., are its roots.

If $\frac{1}{y}$ is substituted for x, and the equation is reduced to the form

$$0 = y^n - \frac{y^{n-2}}{2.3} + \frac{y^{n-4}}{2.3.4.5} - \&c. = \left(y - \frac{1}{\pi}\right)\left(y + \frac{1}{\pi}\right)\left(y - \frac{1}{2\pi}\right)\left(y + \frac{1}{2\pi}\right), \&c. \text{ (Alg. Art. 227.)}$$

Or,

$$y^n\left(1 - \frac{1}{2.3y^2} + \frac{1}{2.3.4.5y^4} - \&c.\right) = y\left(1 - \frac{1}{\pi y}\right) y\left(1 + \frac{1}{\pi y}\right) y\left(1 - \frac{1}{2\pi y}\right) y\left(1 + \frac{1}{2\pi y}\right), \&c.$$

Or, dividing by y^n,

$$1 - \frac{1}{2.3.y^2} + \frac{1}{2.3.4.5.y^4} - \&c. = \left(1 - \frac{1}{\pi y}\right)\left(1 + \frac{1}{\pi y}\right)\left(1 - \frac{1}{2\pi y}\right)\left(1 + \frac{1}{2\pi y}\right), \&c.$$

Or, replacing $\frac{1}{y}$ by x, $\frac{1}{y^2}$ by x^2, &c.

$$1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \&c. = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right), \&c.$$

$$\text{Hence, } \sin. x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right), \&c.$$

Or, substituting $\frac{m}{n} . \frac{\pi}{2}$ for x,

$$\sin. \frac{m}{n} . \frac{\pi}{2} = \frac{m}{n} . \frac{\pi}{2}\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{16n^2}\right)\left(1 - \frac{m^2}{36n^2}\right), \&c.$$

Taking the logarithms of both members of this equation in the system whose base is 10, and denoting the logarithms of this system by the abbreviation, log.

$$\log.\sin.\frac{m}{n}.\frac{\pi}{2}=\log.\pi+\log.\frac{m}{2n}+\log.\left(1-\frac{m^2}{4n^2}\right)+\log.\left(1-\frac{m^2}{16n^2}\right)+\&c.$$

or, since $\log.\frac{m}{2n}=\log.m-\log.2-\log.n$, and since $1-\frac{m^2}{4n^2}=\frac{4n^2-m^2}{4n^2}=\frac{(2n+m)(2n-m)}{2^2n^2}$,

$$\therefore \log.\left(1-\frac{m^2}{4n^2}\right)=\log.(2n+m)+\log.(2n-m)-2\log.2-2\log.n,$$

$\therefore$ by substitution,

$$\log.\sin.\frac{m}{n}.\frac{\pi}{2}=\log.\pi+\log.m-3\log.2-3\log.n+\log.(2n+m)+\log.(2n-m),$$

$$+\log.\left(1-\frac{m^2}{4^2n^2}\right)+\log.\left(1-\frac{m^2}{6^2n^2}\right)+\&c.$$

Now, $l_\varepsilon\left(1-\frac{m^2}{4^2n^2}\right)=-\left(\frac{m^2}{4^2n^2}+\frac{m^4}{2.4^4n^4}+\frac{m^6}{3.4^6n^6}+\&c.\right)$. (App. Art. 10.)

$$\log.\left(1-\frac{m^2}{4^2n^2}\right)=-\mathrm{M}\left(\frac{m^2}{4^2n^2}+\frac{m^4}{2.4^4n^4}+\frac{m^6}{3.4^6n^6}+\&c.\right). \text{ (Alg. Art. 213.)}$$

Similarly, $\log.\left(1-\frac{m^2}{6^2n^2}\right)=-\mathrm{M}\left(\frac{m^2}{6^2n^2}+\frac{m^4}{2.6^4n^4}+\frac{m^6}{3.6^6n^6}\right)+\&c.$

$$\log.\left(\frac{1-m^2}{8^2n^2}\right)=-\mathrm{M}\left(\frac{m^2}{8^2n^2}+\frac{m^4}{2.8^4n^4}+\frac{m^6}{3.8^6n^6}+\&c.\right)$$

. . . . .

In the same manner are found

$$\mathrm{Cos.}\frac{m}{n}.\frac{\pi}{2}=\left(1-\frac{m^2}{n^2}\right)\left(1-\frac{m^2}{3^2n^2}\right)\left(1-\frac{m^2}{5^2n^2}\right),\ \&c.$$

$$\log.\cos.\frac{m}{n}.\frac{\pi}{2}=\log.\left(1-\frac{m^2}{n^2}\right)+\log.\left(1-\frac{m^2}{3^2n^2}\right)+\log.\left(1-\frac{m^2}{5^2n^2}\right)+\&c.$$

$$\log.\cos.\frac{m}{n}.\frac{\pi}{2}=\log.(n-m)+\log.(n+m)-2\log.n+\log.\left(1-\frac{m^2}{3^2n^2}\right).$$

$$+\log.\left(1-\frac{m^2}{5^2n^2}\right)+\&c.$$

$$\log.\left(1-\frac{m^2}{3^2n^2}\right)=-\mathrm{M}\left(\frac{m^2}{3^2n^2}+\frac{m^4}{2.3^4n^4}+\frac{m^6}{2.3^6n^6}+\&c.\right)$$

$$\log.\left(1-\frac{m^2}{5^2n^2}\right)=-\mathrm{M}\left(\frac{m^2}{5^2n^2}+\frac{m^4}{2.5^4n^4}+\frac{m^6}{2.5^6n^6}+\&c.\right)$$

. . . . .

Now, in the series which expresses $\log.\sin.\frac{m}{n}.\frac{\pi}{2}$, it is evident that the terms $\frac{m^2}{4^2n^2}$, $\frac{m^2}{6^2n^2}$, $\frac{m^2}{8^2n^2}$, &c., may be written

$\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\&c.\right)\frac{m^2}{n^2}$; whence, since all the terms of this part of the expression are multiplied by $-\text{M}$, the coefficient of $\frac{m^2}{n^2}$ is $-\text{M}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\&c.\right)$.

Similarly,

the coefficient of $\frac{m^4}{n^4}$ is $-\frac{1}{2}\text{M}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\frac{1}{10^4}+\&c.\right)$

. . $\frac{m^6}{n^6}$ is $-\frac{1}{3}\text{M}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\frac{1}{10^6}+\&c.\right)$

. . $\frac{m^8}{n^8}$ is $-\frac{1}{4}\text{M}\left(\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\frac{1}{10^8}+\&c.\right)$

. . . . .

The value of the series $\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+$ &c., is .1612335167120566 +, and the product of this number by M, (or .43429448 +, Alg. Art. 214.), is .070022826605901 +: this is the value of the coefficient of $\frac{m^2}{n^2}$.

The coefficient of $\frac{m^4}{n^4}$, found in the same manner, is .001117266441661

. . $\frac{m^6}{n^6}$, ...000039229146453

. . . . .

The value of log. sin. $\frac{m}{n}\cdot\frac{\pi}{2}$, contains two terms which may be expressed by their logarithmic values: viz. log. π and $-$ 3 log. 2, or $-$ log. 8. The 10 which is added to the log. sines to render their characteristics positive may be combined with log. π and $-$ log. 8.

From 10 + log. π = 10.497149872694134,
take log. 8 = .903089986991944,
$\therefore$ 10 + log. π $-$ 3 log. 2 = 9.594059885702190.

Substituting this value for 10 + log. π $-$ 3 log. 2, the values of the coefficients of $\frac{m^2}{n^2}$, $\frac{m^4}{n^4}$, &c., for the numerical series $-\text{M}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c\right)$,

$-\frac{1}{2}\text{M}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\text{ \&c.}\right)$, &c., and $\frac{m}{n}\,90^\circ$ for $\frac{m}{n}\cdot\frac{\pi}{2}$;

$$\text{Log. sin. }\frac{m}{n}.\,90^\circ=\log.m.+\log.(2n-m)+\log.(2n+m)-3\log.n.$$

$$+\ 9.594059885702190\,.. \qquad . \qquad .\ (a).$$

$$-\ .070022826605901\,..\times\frac{m^2}{n^2} \qquad .\ (b).$$

$$-\ .001117266441661\,..\times\frac{m^4}{n^4} \qquad .\ (c).$$

$$-\ .000039229146453\,..\times\frac{m^6}{n^6} \qquad .\ (d).$$

$$-\ .000001729270798\,..\times\frac{m^8}{n^8} \qquad .\ (e).$$

$$-\ .000000084362986\,..\times\frac{m^{10}}{n^{10}}$$

$$-\ .000000004348715\,..\times\frac{m^{12}}{n^{12}}$$

$$-\ .000000000231931\,..\times\frac{m^{14}}{n^{14}}$$

$$-\ .000000000012659\,..\times\frac{m^{16}}{n^{16}}$$

$$-\ .000000000000702\,..\times\frac{m^{18}}{n^{18}}$$

$$-\ .000000000000039\,..\times\frac{m^{20}}{n^{20}}$$

. . . . .

From the expression,

$$\log.\cos.\frac{m}{n}.\frac{\pi}{2}=\log.(n-m)+\log.(n+m)-2\log.n,$$

$$-\ \text{M}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{ \&c}\right)\frac{m^2}{n^2},$$

$$-\ \tfrac{1}{2}\text{M}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{ \&c.}\right)\frac{m^4}{n^4},$$

$$\tfrac{1}{3}\ \text{I}\ \left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{ \&c.}\right)\frac{m^6}{n^6},$$

. . . . .

may be obtained, in nearly the same manner, the result following:

$$
\begin{aligned}
\text{Log. cos. } \frac{m}{n}\,90^\circ = {} & \log.(n-m) + \log.(n+m) - 2\log. n. \\
& + 10.000000000000000. \\
& - \quad .101494859341892.. \times \frac{m^2}{n^2} \\
& - \quad .003187294065451.. \times \frac{m^4}{n^4} \\
& - \quad .000209485800017.. \times \frac{m^6}{n^6} \\
& - \quad .000016848348597.. \times \frac{m^8}{n^8} \\
& - \quad .000001480193986.. \times \frac{m^{10}}{n^{10}} \\
& - \quad .000000136502272.. \times \frac{m^{12}}{n^{12}} \\
& - \quad .000000012981715.. \times \frac{m^{14}}{n^{14}} \\
& - \quad .000000001261471.. \times \frac{m^{16}}{n^{16}} \\
& - \quad .000000000124567.. \times \frac{m^{18}}{n^{18}} \\
& - \quad .000000000012456.. \times \frac{m^{20}}{n^{20}} \\
& - \quad .000000000001258.. \times \frac{m^{22}}{n^{22}} \\
& - \quad .000000000000128.. \times \frac{m^{24}}{n^{24}} \\
& - \quad .000000000000013.. \times \frac{m^{26}}{n^{26}}
\end{aligned}
$$

. . . . . .

The log. sines and cosines may be computed from these tables to 15 places of decimals, provided the logarithms of the numbers m, $2n-m$, $2n+m$, &c., are taken to fifteen places.*

Since the value of $\frac{m}{n}$ need not exceed $\frac{1}{2}$, and the value of

---

* The logarithms of numbers, to 20 places of decimals, may be found by means of Tables I., II., III., given at the end of Gardiner's Tables of Logarithms: edit. Lond. 1742.

n, in seconds, is $90 \times 60 \times 60 = 324000$, therefore m is not greater than 162000; $2n - m$ not greater than 486000; $2n + m$ not greater than 810000, &c. The values of these consequently consist of not more than six figures.

54. The log. sines of the tables in common use do not contain more than six or seven places of decimals. To have this degree of approximation, it is not necessary to take the logarithms of m, $2n-m$, $2n+m$, &c., or the coefficients of $\frac{m^2}{n^2}$, $\frac{m^4}{n^4}$, &c., to more than six or seven figures, except in finding the log. sines of very small arcs, when it will be convenient to employ a greater number of figures than seven.

As an example of the application of these tables to the calculation of the log. sine of an angle, let it be required to find the log. sine of 9°, to ten places of decimal figures.

In this instance, since $\frac{m}{n} 90° = 9°$, $\frac{m}{n} = \frac{1}{10}$ $\therefore m = 1$, $n = 10$;

$2n - m = 19$; $2n + m = 21$; $3 \log. n = 3 \log. 10$; $\frac{m^2}{n^2} = \frac{1}{100}$, &c.

Log. m = log. 1 . . . . $= 0$
Log. $(2n-m)$ = log. 19 . . . . . $= 1.2787536009$
Log. $(2n+m)$ = log. 21 . . . . . $= 1.3222192947$
(a) (table, page 85.) . . . . . . $= 9.5940598857$
12.1950327813 (A)
$-3 \log. n = -3 \log. 10$ . . . . $= -3.0000000000$
$-.07002282 + . \times \frac{1}{100}$ . . . . . $= -0.0007002282$
$-.001117 .. \times \frac{1}{10000}$ . . . . . . $= -0.0000001117$
$-.000039 ... \times \frac{1}{1000000}$ . . . . $= -0.0000000000$
-3.0007003399 (B)

The diff. of the absolute numbers marked (A), (B) is 9.1943324414: and this is the log. sin. 9° to ten places of decimal figures.

55. Since $\tan. A = \frac{\sin. A}{\cos. A}$; $\log. \tan. A = \log. \left(\frac{\sin. A}{\cos. A}\right) = \log. \sin. A - \log. \cos. A$:

and since $\cot. A = \frac{\cos. A}{\sin. A}$, $\log. \cot. A = \log. \left(\frac{\cos. A}{\sin. A}\right) = \log. \cos. A - \log. \sin. A$.

The log. tangents and log. cot. may consequently be obtained from the log. sines and log. cosines, by subtraction.

a. As the log. sines of the trigonometrical tables may be found by taking the logarithms of the natural sines, and adding 10 to the characteristics, so the natural sines may be found from the log. sines of the trigonometrical tables, by subtracting 10 from the characteristics, and taking the natural numbers which belong to the remainders, considered as the logarithms of numbers.

b. The logarithmic tables in common use are, —

Sherwin's, which contain the logarithms of numbers from 1 to 101000; and the natural and logarithmic sines, tangents, secants, and versed sines to every minute of the quadrant.

Gardiner's, which contain the logarithms of numbers from 1 to 101000, and the logarithmic sines and tangents to every ten seconds of the quadrant.

Taylor's, which contain the logarithms of numbers from 1 to 101000, and of sines and tangents to every second of the quadrant.

Hutton's, which contain the logarithms of numbers from 1 to 108000, and the natural and logarithmic sines, tangents, and secants to every minute of the quadrant; together with other useful tables.

Callet's, which contain the logarithms of numbers from 1 to 108000, and of sines and tangents to every ten seconds of the quadrant.

Babbage's logarithms of numbers.

The logarithms in these tables are to seven places of figures.

Lalande's logarithms of numbers from 1 to 10000, and of sines and tangents to every minute of the quadrant; to five places of figures.

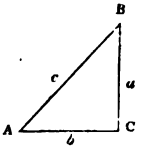
Farley's logarithms of numbers from 1 to 10000, and of sines and tangents to every minute of the quadrant, to six places of figures. The calculations, in this treatise, have been made with Farley's tables of six-figure logarithms.

Dodson's Antilogarithmic Canon, which contains all logarithms under 10000, and their corresponding natural numbers, to eleven places of figures, with differences and proportional parts. In these tables, which are intended for the more readily finding the number corresponding to any given logarithm, the arrangement is in the order contrary to that used in the common tables of numbers and logarithms; the exact logarithms being placed first, and their corresponding nearest numbers in the columns opposite to them.

SECTION V.

RESOLUTION OF PLANE RECTILINEAL TRIANGLES.

56. The relations between the sides and angles of right-angled rectilineal triangles are immediately deducible from the definitions of Article 17; for if ABC is a triangle having the angle C a right angle,



$$\frac{a}{c} = \text{sin. A}, \therefore a = c \text{ sin. A};$$

$$\frac{b}{c} = \text{cos. A}, \therefore b = c \text{ cos. A};$$

$$\frac{a}{b} = \text{tan. A}, \therefore a = b \text{ tan. A};$$

Similarly, $b = c$ sin. B; $a = c$ cos. B; $b = a$ tan. B.

Hence, in a right-angled triangle, either of the sides which contain the right angle is equal

1st. To the hypotenuse* multiplied by the sine of the angle which is opposite to that side; or,

2d. To the hypotenuse multiplied by the cosine of the angle which is adjacent to that side; or,

3d. To the other side multiplied by the tangent of the angle which is opposite to the first side.

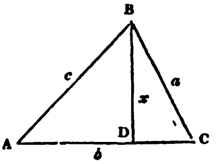
a. Since c sin. A $= a$, and c cos. A $= b$, therefore

$$c = \frac{a}{\text{sin. A}}, \text{ and } c = \frac{b}{\text{cos. A}}.$$

Consequently the hypotenuse is equal to the perpendicular divided by the sine of the opposite acute angle; or to the base divided by the cosine of the adjacent acute angle.

57. To find the relations which subsist between the sides and angles of any rectilineal triangle.

* In right-angled triangles, the side opposite the right angle is named hypotenuse; and of the sides containing the right angle, that which is horizontal is sometimes named base, and that which is vertical, perpendicular.



Let ABC be any triangle; and from B, one of the angles, let BD or x be drawn perpendicular to AC.

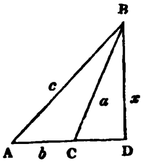
Then,

by Art. 17., $\frac{x}{c} = \sin. A$, and $\frac{x}{a} = \sin. C$;

$$\therefore \frac{x}{c} \div \frac{x}{a} = \sin. A \div \sin. C, \text{ or } \frac{a}{c} = \frac{\sin. A}{\sin. C}.$$

$\therefore a : c :: \sin. A : \sin. C.$ (Alg. Art. 181.)

Similarly, $a : b :: \sin. A : \sin. B.$
$b : c :: \sin. B : \sin. C.$



If the angle C is obtuse, $\frac{x}{a} = \sin. BCD.$

But $\sin. BCD = \sin. BCA = \sin. C.$ (Art. 18. *b.*)

Whence in this case also, $\frac{x}{a} = \sin. C.$

$$\text{Now, } \frac{x}{c} = \sin. A, \therefore \frac{x}{c} \div \frac{x}{a} = \sin. A \div \sin. C, \text{ or } \frac{a}{c} = \frac{\sin. A}{\sin. C};$$

or $a : c :: \sin. A : \sin. C.$

Consequently, in every rectilineal triangle, the sides are proportional to the sines of the angles which are opposite to them.

a. Since $a : b :: \sin. A : \sin. B.$

$$\therefore \frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B}. \quad \text{(Alg. Art. 181. } f.)$$

$$\text{But } \frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}. \quad \text{(Art. 40.)}$$

$$\therefore \frac{a+b}{a-b} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}.$$

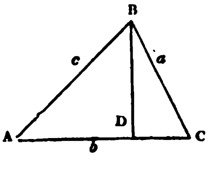
or, $a + b : a - b :: \tan. \frac{1}{2}(A+B) : \tan. \frac{1}{2}(A-B).$*

Hence the sum of any two sides of a triangle, is to their difference, as the tangent of half the sum of the angles opposite to those sides, to the tangent of half their difference.

---

* If $b > a$, $B > A$, and the second and fourth terms of the proportion are $b - a$, $\tan. \frac{1}{2}(B - A)$.

b. In the triangle ABC, let BD be drawn perpendicular to AC:



then $\mathrm{CD}=a\cos.\ c$, and $\mathrm{DA}=c\cos.\ \mathrm{A}$. (Art. 56.)

$\therefore\ \mathrm{CD}+\mathrm{DA}=\mathrm{AC}=a\cos.\ \mathrm{C}+c\cos.\ \mathrm{A}$,

or $b=a\cos.\ \mathrm{C}+c\cos.\ \mathrm{A}$, (1):

similarly, $a=b\cos.\ \mathrm{C}+c\cos.\ \mathrm{B}$, (2)

$c=a\cos.\ \mathrm{B}+b\cos.\ \mathrm{A}$, (3).

Multiplying the terms of (2) by a, $a^2=ab\cos.\ \mathrm{C}+ac\cos.\ \mathrm{B}$, (4)

(1) by b, $b^2=ab\cos.\ \mathrm{C}+bc\cos.\ \mathrm{A}$, (5)

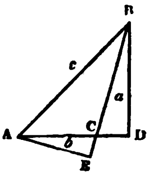
(3) by c, $c^2=ac\cos.\ \mathrm{B}+bc\cos.\ \mathrm{A}$, (6).

Subtracting (6) from (4)+(5), $a^2+b^2-c^2=2ab\cos.\ \mathrm{C}$, (7)

(5) from (4)+(6), $a^2+c^2-b^2=2ac\cos.\ \mathrm{B}$, (8)

(4) from (5)+(6), $b^2+c^2-a^2=2bc\cos.\ \mathrm{A}$, (9).

If C is obtuse, $b=\mathrm{AD}-\mathrm{CD}$:



$\mathrm{AD}=c\cos.\ \mathrm{A}$; $\mathrm{CD}=a\cos.\ \mathrm{BCD}=a\times-\cos.\ \mathrm{BCA}$; (Art. 24. *a*.); $=-a\cos.\ \mathrm{C}$;

$\therefore\ b=c\cos.\ \mathrm{A}-(-a\cos.\ \mathrm{C})=c\cos.\ \mathrm{A}+a\cos.\ \mathrm{C}=a\cos.\ \mathrm{C}+c\cos.\ \mathrm{A}$.

Similarly, $a=c\cos.\ \mathrm{B}-(-b\cos.\ \mathrm{C})=b\cos.\ \mathrm{C}+c\cos.\ \mathrm{B}$.

From (9) is obtained by transposition, $a^2=b^2+c^2-2bc\cos.\ \mathrm{A}$: a result which may be thus expressed in words:—The square of a side of any triangle is equal to the sum of the squares of the two other sides, less, twice the product of these two sides multiplied by the cosine of the angle included between them.

c. With formulæ (7), (8), (9), and proper data, the unknown sides and angles of any triangles may be determined. For example, if a, b, C, are given to find c,

$$\text{by (7), } c=\sqrt{a^2+b^2-2ab\cos.\ \mathrm{C}};$$

Or, if b, c, a, are given to find A, by (9),

$$\cos.\ \mathrm{A}=\frac{b^2+c^2-a^2}{2bc}:$$

but the results are not adapted to calculation by logarithms.

Expressions which give the sine, cosine, or tangent of any angle of a triangle in terms of the three sides of that triangle, and which are suitable for logarithmic calculation, can, however, be obtained as follows.

d. Since $2\sin.^2\frac{\mathrm{A}}{2}=1-\cos.\ \mathrm{A}$, (Art. 35.)

and . . . $\cos.\ \mathrm{A}=\frac{b^2+c^2-a^2}{2bc}$, (Art. 57. *c*.)

$\therefore 2\sin.^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2bc}$: whence, by successive reductions,

$$2\sin.^2\frac{A}{2}=\frac{2bc-b^2-c^2+a^2}{2bc}=\frac{a^2-(b^2-2bc+c^2)}{2bc}=\frac{a^2-(b-c)^2}{2bc}.$$

Now,

$$a^2-(b-c)^2=(a+c-b)(a+b-c)=2(s-b)2(s-c)=4(s-b)(s-c).\ (\text{Art. } 8.\ a.)$$

$$\therefore\ 2\sin.^2\frac{A}{2}=\frac{4(s-b)(s-c)}{2bc},\text{ and }\sin.^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

$$\left.\begin{aligned}\therefore\ \sin.\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{bc}}.\\ \text{Similarly,}\qquad \sin.\frac{B}{2}&=\sqrt{\frac{(s-a)(s-c)}{ac}};\\ \sin.\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{ab}}.\end{aligned}\right\}\text{A.}$$

Let A, A′ be arcs of circles containing the same number of degrees, and let 1, r, be their respective radii: then

$$\sin.^2\tfrac{1}{2}A=\frac{\sin.^2\frac{1}{2}A'}{r^2},\ (\text{Art. } 30.\ l.)$$

$$\therefore\ \frac{\sin.^2\frac{1}{2}A'}{r^2}=\frac{(s-b)(s-c)}{bc},\text{ or } bc:(s-b)(s-c)::r^2:\sin.^2\tfrac{1}{2}A'.$$

In words: The rectangle contained by two sides of any triangle, is to the rectangle contained by the excesses of the semiperimeter over these sides, as the square of the radius, is to the square of the sine of half the angle included by the two sides.

e. Since $1=1$

and $\cos.A=\frac{b^2+c^2-a^2}{2bc}$, (Art. 57. *c.*)

$$\therefore 1+\cos.A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{b^2+2bc+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}.$$

Now, $1+\cos.A=2\cos.^2\frac{A}{2}$, (Art. 35.);

And $(b+c)^2-a^2=(a+b+c)(b+c-a)=2s\times 2(s-a)=4s(s-a)$. (Art. 8. *a.*)

Substituting these values instead of $1+\cos.A$ and $(b+c)^2-a^2$;

$$2\cos.^2\frac{A}{2}=\frac{4s(s-a)}{2bc}=\frac{2s(s-a)}{bc};$$

$$\therefore\ \cos.^2\frac{A}{2}=\frac{s(s-a)}{bc},$$

$$\left.\begin{aligned}\text{and } \cos.\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}\\ \text{Similarly, } \cos.\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ac}}\\ \cos.\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}\end{aligned}\right\} B.$$

If $\frac{1}{2}A$, $\frac{1}{2}A'$ are arcs of circles containing the same number of degrees, and 1, r, their respective radii,

$$\cos.^2\tfrac{1}{2}A = \frac{\cos.^2\frac{1}{2}A'}{r^2} \quad \therefore \frac{\cos.^2\frac{1}{2}A'}{r^2} = \frac{s(s-a)}{bc}, \text{or, } bc : s(s-a) :: r^2 : \cos.^2\tfrac{1}{2}A'.$$

Whence this theorem: The rectangle contained by two sides of any triangle, is to the rectangle contained by the semiperimeter and its excess above the third side, as the square of the radius, is to the square of the cosine of half the angle included between the two sides.

f. Since $\tan.\frac{A}{2} = \sin.\frac{a}{2} \div \cos\frac{A}{2}$, (Art. 19. *c.*)

$$\text{and } \sin.\frac{A}{2} = \sqrt{\frac{(s-b(s-c)}{bc}} (\text{Art. 57. } d.); \cos.\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \ (\text{Art. 57. } e.)$$

$$\left.\begin{aligned}\therefore \tan.\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\ \text{Similarly, } \tan.\frac{B}{2} &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\\ \tan.\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\end{aligned}\right\} C.$$

$$f'. \text{ Since } \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{s-a}{s-a}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$\therefore \tan.\frac{A}{2} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, or, denoting the radical quantity by M,

$$\tan.\frac{A}{2} = \frac{M}{s-a}. \text{ Similarly, } \tan.\frac{B}{2} = \frac{M}{s-b}; \ \tan.\frac{C}{2} = \frac{M}{s-c} \quad . \quad . \quad C'$$

If $\tan.\frac{1}{2}\frac{A'}{r}$ is substituted for $\tan.\frac{1}{2}\frac{A}{1}$ in the first of the three equations marked C, and both members of the equation are squared,

$$\frac{\tan.^2\frac{1}{2}A'}{r^2} = \frac{(s-b)(s-c)}{s(s-a)}$$

$$\text{or } s(s-a) : (s-b)(s-c) :: r^2 : \tan.^2\tfrac{1}{2}A'.$$

Whence this other theorem: The rectangle contained by the semiperimeter of any triangle and its excess above one of the sides, is to the rectangle contained by the excesses of the semiperimeter above the other two sides, as the square of the radius, is to the square of the tangent of half the angle contained by the two sides.

g. The expression $b^2+c^2-a^2=2bc \cos. \text{A}$, or $\cos. \text{A}=\frac{b^2+c^2-a^2}{2bc}$, may be immediately obtained from the value of the perpendicular, y, in Art. 8. *a*., and from the definition of the cosine:

$$\text{For } y=\frac{b^2+c^2-a^2}{2b}, \text{ and } \frac{y}{c}=\cos. \text{A}.$$

$$\therefore \cos. \text{A}=\frac{b^2+c^2-a^2}{2bc}.$$

A formula by which $\sin. \text{A}$ is expressed in terms of a, b, c, and which is suitable for logarithmic calculation may also be found from the same article:

$$\text{For } \frac{bx}{2}=\sqrt{s(s-a)(s-b)(s-c)},$$

$$\therefore x=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\text{And } \frac{x}{c}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

But from the definition of the sine $\frac{x}{c}=\sin. \text{A}$,

$$\therefore \sin. \text{A}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \quad . \quad . \quad (1.)$$

$$\text{Similarly,} \quad \sin. \text{B}=\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)} \quad . \quad . \quad (2.)$$

$$\sin. \text{C}=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} \quad . \quad . \quad (3.)$$

If the expression marked (1.) is divided by that marked (2),

$$\sin. \text{A} \div \sin. \text{B}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \div \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

or $\frac{\sin. \text{A}}{\sin. \text{B}}=\frac{a}{b}$, which is the first formula of this Article.

57′. The formulæ of Articles 56. and 57., are sufficient for the calculation of the numerical values of the sides and angles of all rectilineal triangles, from proper data. The quantities which are made to enter into these calculations are the logarithms of the numerical values of the sides and the logarithmic sines, cosines, &c. of the angles.

Every triangle has three sides and three angles; the number of these quantities is therefore six: and the general problem in the resolution of triangles is — Having given three of these parts, to find a fourth. Of the three parts given, one at the least must be a side; for if the three angles alone are given, the ratios of the sides can be determined, but not their numerical values. (Euc. vi. 4.)

In particular cases more parts than three are given; for if in a right-angled triangle one side and an acute angle are given, it is evident that the third angle, which is the complement of the given acute angle, is also given; and if in an oblique-angled triangle, a side and two angles are given, that the third angle, which is the supplement of the two given angles, is also given.

The calculations are made with logarithms; for which reason the quantities involved are connected by the processes of multiplication and division, which are suited to calculation by logarithms, rather than by the processes of addition and subtraction, which must be performed with natural numbers.

a. The tabular logarithm of a sine, cosine, &c., is equal to the real logarithm of that sine, or cosine, &c., augmented by 10. Hence, for every log. sine, &c. in either member of an equation, if connected by the sign +, it is necessary to subtract 10 from that member, or to add 10 to the other member: and if connected by the sign —, to add 10 to the member in which the log. sine, &c. is, or to subtract 10 from the other member. If there are log. sines, &c. in both members of an equation, or with contrary signs, in the same member, the *tens*, or some of them, may compensate each other: such should be struck out.

58. Resolution of right-angled triangles.*

The cases of right-angled triangles may be reduced to four, viz.

1st. Given the hypotenuse c, and an acute angle, A; to find the angle B, the perpendicular a, and the base b.

2d. Given the base b, and the acute angle A; to find the angle B, the perpendicular a, and the hypotenuse c.

3d. Given the hypotenuse c, and the base b, to find the angles A, B, and the perpendicular a.

4th. Given the perpendicular a, and the base b, to find the angles A, B, and the hypotenuse c.

CASE I. Given c, A: required B, a, b.

1st. $B = 90° - A$. (Art. 15.)

2d. $a = c \sin. A$. (Art. 56.)

$\therefore \log. a = \log. c + \log. \sin. A - 10$. (Art. 57'. *a.*)

3d. $b = c \cos. A$.

$\therefore \log. b = \log. c + \log. \cos. A - 10$.

If B is given instead of A, then $A = 90° - B$.

and, as before, $a = c \sin. A$; $b = c \cos. A$: or, employing the given angle B,

$$a = c \cos. B; \; b = c \sin. B.$$

---

* For the resolution of triangles by construction, see Appendix, Art. 11.

Examples.—1st. Let $c = 361.4$ yds. and $A = 41^\circ\,12'$: required B, a, b.

$$\text{Log. } a = \left\{\begin{array}{l} \text{Log. } 361.4 \quad . \quad . \quad . \quad . \\ +\text{log. sin. } 41^\circ\,12' \quad . \quad . \quad . \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right. \begin{array}{r} 2.557988 \\ +9.818681 \\ -10.000000 \\ \hline 2.376669 \end{array}$$

The number of this logarithm is 238.05,

$$\therefore a = 238.05 \text{ yards.}$$

$$\text{Log. } b = \left\{\begin{array}{l} \text{log. } 361.4 \quad . \quad . \quad . \quad . \\ +\text{log. cos. } 41^\circ\,12' \quad . \quad . \quad . \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right. \begin{array}{r} 2.557988 \\ 9.876457 \\ -10.000000 \\ \hline 2.434445 \end{array}$$

The number of this log. is 271.92,

$$\therefore b = 271.92 \text{ yards.}$$

$$B = 90^\circ - 41^\circ\,12' = 48^\circ\,48',$$

Whence $B = 48^\circ\,48'$; $a = 238.05$ yards; $b = 271.92$ yards.

2d. Let $c = 450$ feet, and $B = 52^\circ\,38'$; required A, a, b.
Ans. $A = 37^\circ\,22'$; $a = 273.1$; $b = 357.6$ feet.

3d. Let $c = 4264.3$ yds., and $A = 56^\circ\,29'\,15''$: required B, a, b.
Ans. $B = 33^\circ\,30'\,45''$; $a = 3555$ yds.; $b = 2354$ yds.

CASE II. Given A, b; required B, a, c.

1st. $B = 90^\circ - A$.

2d. $a = b \text{ tan. } A$. (Art. 56.)

$$\therefore \text{log. } a = \text{log. } b + \text{log. tan. } A - 10,$$

3d. $c = \dfrac{b}{\text{cos. } A}$.

$$\therefore \text{log. } c = \text{log. } b - \text{log. cos. } A + 10.$$

or, log. c = log. b + arithmetical complement of log. cos. $A - 10 + 10$. (Arith. Art. 42.)

or, log. c = log. b + arith. comp. of log. cos. A.

If B is given instead of A; then $A = 90^\circ - B$; $a = b \text{ tan. } A$; $c = \dfrac{b}{\text{cos. } A}$,

$$\text{or, } a = b \text{ cot. } B; \; c = \frac{b}{\text{sin. } B}.$$

Ex.—1st. Let $A = 53^\circ\,8'$; $b = 2880$ yards: required B, a, c.

$$B = 90^\circ - 53^\circ\,8' = 36^\circ\,52'.$$

$$\text{Log. } a = \left\{\begin{array}{l} \text{log. } 2880 \quad . \quad . \quad . \quad . \\ +\text{log. tan. } 53^\circ\,8' \quad . \quad . \quad . \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right. \begin{array}{r} 3.459392 \\ +10.124990 \\ -10.000000 \\ \hline \end{array}$$

The number of this log. is 3840.5 . . . . 3.584382

$$\therefore a = 3840.5 \text{ yards.}$$

$$\text{Log. } c = \left\{ \begin{array}{ll} \text{log. } 2880 \quad . \quad . \quad . \quad . & 3.459392 \\ + \text{arith. comp. of log. cos. } 53^\circ\ 8' & 0.221881 \end{array} \right.$$

The number of this log. is 4800.36 . . . 3.681273

$\therefore c = 4800.36$ yards.

Whence $B = 36^\circ\ 52'$; $a = 3840.5$ yards; $c = 4800.36$ yards.

Ex. 2. Given $A = 4^\circ\ 44'$; $a = 694{\cdot}73$: required B, b, c.
Ans. $B = 85^\circ\ 16'$; $b = 8390$; $c = 8419$.

Ex. 3. Given $A = 17^\circ\ 12'\ 51''$; $b = 317{\cdot}6$: required B, a, c.
Ans. $B = 72^\circ\ 47'\ 9''$; $a = 98{\cdot}399$; $c = 332{\cdot}49$.

CASE III. Given b, c, to find A, B, a.

1st. $\text{Cos. } A = \frac{b}{c}$. (Art. 56.)

$$\therefore \text{log. cos. } A = \text{log. } b - \text{log. } c + 10.$$

or, log. cos. $A = \text{log. } b +$ arith. comp. of log. $c - 10 + 10 = \text{log. } b +$ arith. comp. of log. c.

2d. $B = 90^\circ - A$.
3d. $a = c \text{ sin. } A$.

$$\therefore \text{log. } a = \text{log. } c + \text{log. sin. } A - 10.$$

$$\text{Otherwise, } a = \sqrt{c^2 - b^2}. \quad \text{(Euc. I. 47.)}$$

If a, c, are given to find A, B, b.

$$\text{sin. } A = \frac{a}{c}; \quad B = 90^\circ - A; \quad b = c \text{ cos. } A.$$

Ex. 1. Let $b = 473.8$ yards, $c = 643.7$ yards: required A, B, a.

$$\text{Log. cos. } A = \left\{ \begin{array}{ll} \text{log. } 473.8 \quad . \quad . \quad . \quad . & 2.675595 \\ + \text{arith. comp. of log. } 643.7 \quad . & +7.191316 \end{array} \right.$$

The angle of this log. cos. is 42° 36′ 12″ . . 9.866911

$\therefore A = 42^\circ\ 36'\ 12''$

$B = 90 - 42^\circ\ 36'\ 12'' = 47^\circ\ 23'\ 48''$.

$$\text{Log. } a = \left\{ \begin{array}{ll} \text{log. } 643.7 \quad . \quad . \quad . \quad . & 2.808684 \\ + \text{log. sin. } 42^\circ\ 36'\ 12'' \quad . \quad . & +\ 9.830536 \\ -10 \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array} \right.$$

The number of this logarithm is 435.732 . . 2.639220

$\therefore a = 435.732$ yards.

Whence, $A = 42^\circ\ 36'\ 12''$; $B = 47^\circ\ 23'\ 48''$; $a = 435.732$ yards.

Ex. 2. Given $a = 30$; $c = 36{\cdot}06$: required A, B, b.
Ans. $A = 56^\circ\ 18'$; $B = 33^\circ\ 42'$; $b = 20$.

Ex. 3. Given $b = 1277$; $c = 2132$: required A, B, a.
Ans. $A = 53^\circ\ 12'\ 14''$; $B = 36^\circ\ 47'\ 46''$; $a = 1707.2$.

CASE IV. Given a, b: required A, B, c.

1st. $\text{Tan. } A = \frac{a}{b}$. (Art. 56.)

$\therefore \log. \tan. A = \log. a - \log. b + 10 = \log. A + \text{arith. comp. of } \log. b - 10 + 10 = \log. a + \text{arith. comp. } \log. b.$

2d. $B = 90° - A$.

3d. $c = \frac{a}{\sin. A}$. (Art. 56.)

$\therefore \log. c = \log. a - \log. \sin. A + 10 = \log. a + \text{arith. comp. of } \log. \sin. A.$

Ex. 1. Given $a = 43$ yards; $b = 55$ yards: required A, B, c.

$$\text{Log. tan. } A = \begin{cases} \log. 43 \quad . \quad . \quad . \quad . & 1.633468 \\ + \text{arith. comp. of } \log. 55 \quad . \quad . & 8.259637 \end{cases}$$

The angle of this log. tan. is 38° 1′ 8″ . . . 9.893105

$$\therefore A = 38° \, 1' \, 8''.$$

$$B = 90° - 38° \, 1' \, 8'' = 51° \, 58' \, 52''.$$

$$\text{Log. } c = \begin{cases} \log. 43 \quad . \quad . \quad . \quad . \quad . & 1.633468 \\ + \text{arith. comp. of } \log. \sin. 38° \, 1' \, 8'' & 0.210475 \end{cases}$$

The number of this log. is 69.81 . . . . 1.843943

$$\therefore c = 69.81 \text{ yards.}$$

Whence, $A = 38° \, 1' \, 8''$; $B = 51° \, 58' \, 52''$; $c = 69.81$ yards.

Ex. 2. Given $a = 76.5$; $b = 47.2$: required A, B, c.
Ans. $A = 58° \, 19' \, 32''$; $B = 31° \, 40' \, 28''$; $c = 89.889$.

Ex. 3. Given $a = 751.8$; $b = 534.2$: required A, B, c.
Ans. $A = 54° \, 36' \, 14''$; $B = 35° \, 23' \, 46''$; $c = 922.3$.

59. Resolution of oblique-angled triangles.

The cases of oblique-angled triangles may also be reduced to four; viz.

1st. Given a side and two angles to find the remaining angle and sides.

2d. Given two sides and an angle opposite one of these sides, to find the remaining angles and side.

3d. Given two sides and the included angle, to find the remaining angles and side.

4th. Given the three sides to find the three angles.

CASE I. Given A, B, a: required C, b, c.

1st. $C = 180° - (A + B)$. (Art. 15.)

2d. $b = \frac{a \sin. B}{\sin. A}$. (Art. 57.)

$\therefore \log. b = \log. a + \log. \sin. B - 10 - \log. \sin. A + 10;$

or, $\log. b = \log. a + \log. \sin. B - \log. \sin. A;$

or, $\log. b = \log. a + \log. \sin. B + \text{arith. comp. of } \log. \sin. A - 10.$

3d. log. c = log. a + log. sin. C − log. sin. A,
or, log. c = log. a + log. sin. C + arith. comp. of log. sin. A − 10.

When two angles of a triangle are given, the third angle is given (Art. 57.): consequently, in Case I. the given side is always opposite a known angle.

In accommodating the proportions of Article 57. to an example, it is necessary to make the sine of the angle, opposite the given side, the first term of the proportion, the sine of the angle opposite the required side, the second, and the given side, the third: then the required side, which is the fourth term of the proportion, is obtained by multiplying together the second and third terms, and dividing the product by the first; and the log. of the required side, by adding the log. sine of the angle opposite the required side to the log. of the given side, and subtracting the log. sine of the angle opposite the given side from the sum.

Ex. 1. Given $\text{A}=74^\circ\,14'$; $\text{B}=49^\circ\,23'$; $a=471.5$: required C, b, c.
$\text{C}=180^\circ-(74^\circ\,14'+49^\circ\,23')=180^\circ-123^\circ\,37'=56^\circ\,23'$.

$$\text{Log. } b=\left\{\begin{array}{lr}\text{log. } 471.5 \quad . \quad . \quad . \quad . & 2.673482\\ +\text{log. sin. } 49^\circ\,23' \quad . \quad . \quad . & +9.880289\\ +\text{arith. comp. of log. sin. } 74^\circ\,14' & +0.016655\\ -10 \quad . \quad . \quad . \quad . \quad . & -10.000000\end{array}\right.$$

The number of this log. is 371.9 . . . 2.570426

$\therefore b=371.9.$

$$\text{Log. } c=\left\{\begin{array}{lr}\text{log. } 471.5 \quad . \quad . \quad . \quad . & 2.673482\\ +\text{log. sin. } 56^\circ\,23' \quad . \quad . \quad . & +9.920520\\ +\text{arith. comp. of log. sin. } 74^\circ\,14 & +0.016655\\ -10 \quad . \quad . \quad . \quad . \quad . & -10.000000\end{array}\right.$$

The number of this log. is 408. . . 2.610657

$\therefore c=408.$

Whence, $\text{C}=56^\circ\,23'$; $b=371.9$; $c=408$.

Ex. 2. Given $\text{A}=101^\circ\,41'$; $\text{C}=29^\circ\,19'$; $a=96$: required B, b, c.

Ans. $\text{B}=49^\circ$; $b=73.98$; $c=48$.

Ex. 3. Given $\text{B}=79^\circ\,46'\,38''$; $\text{C}=55^\circ\,59'\,58''$; $c=368$: required A, a, b.

Ans. $\text{A}=44^\circ\,13'\,24''$; $a=309.595$; $b=436.844$.

Case II. Given A, a, c, required B, C, b.

1st. $\text{sin. C}=\dfrac{c\ \text{sin. A}}{a}$, (Art. 57.)

$\therefore$ log. sin. C − 10 = log. c + log. sin. A − 10 − log. a,

or, log. sin. C = log. c + log. sin. A − log. a,
or, log. sin. C = log. c + log. sin. A + arith. comp. of log. a − 10.

2d. $B = 180^\circ - (B + C)$.

3d. $\log. b = \log. a + \log. \sin. B +$ arith. comp. of $\log. \sin. a - 10$. (Case I.)

The two supplementary angles C and $180^\circ - C$, of which the one is acute and the other obtuse (except $C = 90^\circ$) have the same sine, (Art. 18. *b*.). Unless, therefore, it is known whether the angle C is acute or obtuse, Case II. is indeterminate.

Now, when $a > c$, $A > C$, (Euc. I. 18.); therefore C is an acute angle, (Euc. I. 17.) and has only one value.

But when $a < c$, $A < C$; and the data give no means of determining whether C or $180^\circ - C$ is the proper value.

If C has two values, B and b also have each two values. (See App. Art. 11. Ex. 3, 4.)

Ex. 1. Given $A = 78^\circ\ 13'$; $a = 153$; $c = 137$: required B, C, b.

$$\text{Log. sin. C} = \left\{ \begin{array}{lr} \text{log. } 137 \quad . \quad . \quad . \quad . & 2.136721 \\ + \text{log. sin. } 78^\circ\ 13' \quad . \quad . \quad . & +9.990750 \\ + \text{arith. comp. of log. } 153 \quad . & +7.815308 \\ -10 \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array} \right.$$

This is the log. sin. of the angle 61° 13′ 46″ . 9.942779

$\therefore C = 61^\circ\ 13'\ 46''$.

Since $c < a$, $C < A$: the angle C in this instance has, therefore, only one value.

$B = 180^\circ - (78^\circ\ 13' + 61^\circ\ 13'\ 46'') = 180^\circ - 139^\circ\ 26'\ 46'' = 40^\circ\ 33'\ 14''$.

$$\text{Log. } b = \left\{ \begin{array}{lr} \text{log. } 153 \quad . \quad . \quad . \quad . & 2.184691 \\ + \text{log. sin. } 40^\circ\ 33'\ 14'' \quad . \quad . & +9.813020 \\ + \text{arith. comp. of log. sin. } 78^\circ\ 13' & +0.009250 \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array} \right.$$

The number of this log. is 101.617. . . . 2.006961

$\therefore b = 101.617$.

Whence, $B = 40^\circ\ 33'\ 14''$; $C = 61^\circ\ 13'\ 46''$; $b = 101.617$.

Ex. 2. Given $A = 41^\circ\ 36'$, $a = 248$; $c = 354$: required B, C, b.

$$\text{Log. sin. C} = \left\{ \begin{array}{lr} \text{log. } 354 \quad . \quad . \quad . \quad . & 2.549003 \\ + \text{log. sin. } 41^\circ\ 36' \quad . \quad . \quad . & +9.822120 \\ + \text{arith. comp. of log. } 248 \quad . & +7.605548 \\ -10 \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array} \right.$$

The angle of this log. sin. is 71° 23′ . . . 9.976671

$\therefore C = 71^\circ\ 23'$.

But, since $a < c$, the angle C may also have the value $180^\circ - 71^\circ\ 23'$ or $108^\circ\ 37'$.

$\therefore B = 180^\circ - (41^\circ\ 36' + 71^\circ\ 23') = 180^\circ - 112^\circ\ 59' = 67^\circ\ 1'$.

or $B = 180^\circ - (41^\circ\ 36' + 108^\circ\ 37') = 180^\circ - 150^\circ\ 13' = 29^\circ\ 47'$.

$$\therefore \text{log. } b = \left\{ \begin{array}{lr} \text{log. } 248 \quad . \quad . \quad . \quad . & 2.394452 \\ + \text{log. sin. } 29^\circ\ 47' \quad . \quad . \quad . & +9.696113 \\ + \text{arith. comp. of log. sin. } 41^\circ\ 36' & +0.177880 \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array} \right.$$

The number of this log. is 185.54. . . . 2.268445

SECTION VI.

MEASUREMENT OF A BASE LINE. — DESCRIPTION OF INSTRUMENTS EMPLOYED IN THE MENSURATION OF HEIGHTS AND DISTANCES, AND IN SURVEYING AND LEVELLING.

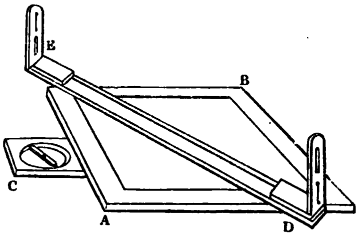
MEASUREMENT OF A BASE LINE.

60. The manner of measuring on the ground the length of a base line with the accuracy necessary for a trigonometrical survey of an extensive region, is described in the treatise of Practical Astronomy and Geodesy (Art. 391.); but, for an ordinary survey, a base is frequently measured in the following manner. On a road, or on a piece of ground nearly level, a white cord or tape at least 100 feet long is stretched, horizontally if possible, in any convenient direction between two pickets of wood, or two iron pins, driven into the ground; the length of the cord is then measured by means of a rod 10 or 20 feet long, which is applied successively along it, the places of the extremities of the rod being indicated on the cord, as the measurement proceeds, by marks, sometimes by small pins thrust into it, in order, when the position of the rod is changed, to allow its rear extremity to be placed as accurately as possible at the point which the head had previously occupied. When nearly the whole length of the cord has been measured, the rear picket, with the end of the cord which is attached to it, is taken up, and being carried forward beyond the other picket, the cord is again stretched in the line of its former direction — that line having been determined by means of other pickets previously planted for the purpose: the measurement is then continued from the point where the head of the rod was last situated, its place having been preserved by means of a pointed wire, or arrow, pressed vertically into the ground. The position of the cord may be changed in like manner as often as necessary, till the whole length of the base has been measured.

If the measured line should not be precisely horizontal, its inclination must be found; and it may then be reduced to that position by multiplying it into the cosine of the elevation or depression.

OF THE PLANE TABLE.

61. This simple instrument, with which, when great accuracy is not required, a survey of ground may be made with considerable facility, consists of a board, as A B, varying in form and magnitude from a square each of whose sides is 8 inches in length, to a rectangle 16 inches long and 12 inches broad: it is supported on a small brass pillar, the lower part of which turns in a socket at the top of a stand with three legs; and, being placed as nearly as possible in a horizontal position, it is capable of being turned round in that position upon the axis of the pillar. On one side of the board is a compass box, C, and by means of the needle in the latter, either of the pairs of parallel edges may be placed in the direction of the meridian, or at any angle with it.



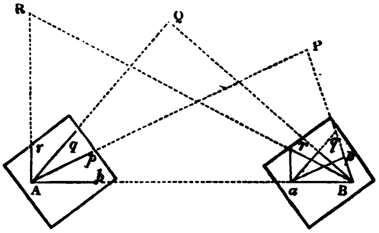
A sheet of paper is attached to the board, frequently by paste at the four edges, but some plane tables are provided with a square or rectangular frame, which, by means of a rebate, is made to confine the paper, while its upper surface is in the same plane as that of the board. The upper surface of the frame is frequently divided into 360 parts forming a scale of tangents of degrees, on the four sides, the middle of the board being considered as the centre of the circle, and the degrees serving to express the value of an observed angle: in some cases, a scale of inches decimally divided is placed on each of the four sides, and, by these, lines parallel to one another may be drawn on the paper.

An alidade or bar, DE, of wood or metal equal in length to a diagonal of the board, serves as a ruler for drawing lines, on which account one of its edges is chamfered. By pressing into the board, at any point, a fine needle, to represent the point in the ground over which the table is placed, and bringing the chamfered edge against it, the observer can turn the ruler about the needle as an axis: then, looking along the edge, he may direct the latter to any distant object; and drawing a line in this position of the ruler, the table being prevented from turning round, by the tightening of a screw which is provided for the purpose, the direction of the object from the observer is represented. In general, each end of the ruler is furnished with a brass plate, which, when the instrument is being used, is placed in a vertical position; each plate is perforated to serve as a sight vane, and the ruler is to be turned upon the needle till the object is seen on looking through both perforations; the edge of the ruler is then parallel to a visual ray from the object to the eye.

On some plane tables a certain point is taken in a line drawn through the middle point, parallel to two sides and equally distant from that middle point and one of the other two sides; at this point is a small hole in a piece of ivory which is let into the wood; and the margin of the frame is divided in tangents of degrees from the point as a centre: the degrees are subdivided into spaces of 15 or 10 minutes each; and the divisions, as well as the 360 degrees before mentioned, may be used to obtain the values of angles subtended between objects. Since these appertain to a circle of longer radius, they have some advantage over the others in giving the measure of an observed angle more accurately.

The plane table is used in various ways. If it be required to lay down the outline of an enclosed piece of ground of small dimensions, from some point in the interior of which all the angular points are visible, the instrument may be set up at that point; and the board being laid as nearly as possible in a horizontal position, either by a small spirit level, or simply by adjusting it so that the compass needle may traverse on its pivot in the box, also two sides of the board being placed, by means of the needle, in any given direction (suppose parallel to the magnetic or to the true meridian), let a needle be pressed into any convenient part of the board, to represent the point on the ground immediately under the table: about the needle as an axis, turn next the edge of the ruler successively to the different angular points of the boundary, and draw lines on the board in those directions; then, if the distances of those angular points from the needle in the board be measured, and laid down on the corresponding lines by any convenient scale, the points being joined by lines, there will be obtained the required outline of the enclosure.

In general, however, the remarkable objects on the ground to be surveyed are represented on paper in their relative positions, as if projected on a horizontal plane, by a triangulation: thus, a base being measured on the ground in feet, yards, or chains, a line is drawn on the plane table (when that instrument is used), and, by any convenient scale, is made equal to that measured length: then, the instrument being placed successively at the extremities of the base, lines are drawn from the corresponding extremities of the base line on the table in the directions of the objects, and their intersections determine the places of those objects.



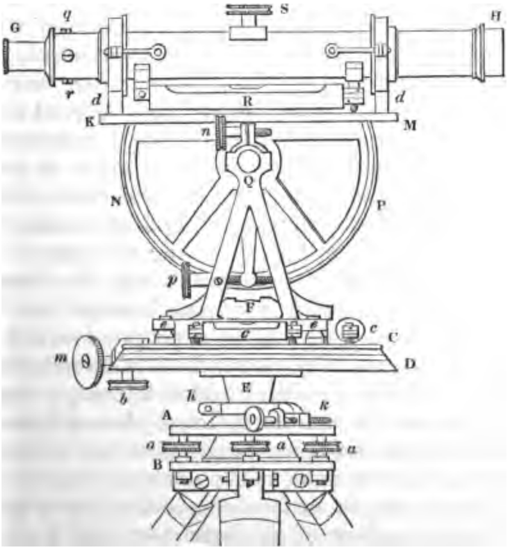
Let a line from A to B be the base actually measured, and let A on the table represent the point A on the ground vertically above which the former point is situated; then, the edge of the alidade or ruler being held on the line A*b*, which has been drawn at pleasure on the table to represent AB, let the whole table be turned round horizontally, till the edge of the ruler, if produced, would meet a staff set up at B on the ground. The table is now screwed to its stand so as to be incapable of turning round, and the ruler alone is to be turned upon a needle at A till its edge tends successively in the directions of the objects P, Q, R, &c., when indefinite lines A*p*, A*q*, A*r*, &c. are drawn on the paper in those directions. The instrument is then removed, and set up so that the point *b* may be vertically above B on the ground: it is next turned round horizontally as before, till a ruler laid on B*a* lies in the direction BA; and then, the board being prevented from turning, the ruler is placed successively in the positions BP, BQ, BR, &c. Lines are lastly drawn on the table in these directions till they intersect the former lines in *p*, *q*, *r*, &c., and these represent on the paper the places of the several objects P, Q, R, &c.

The compass has not been used in this operation; but it is easy to see that if, while the table was at A, the degree indicated by the needle on the rim of the box were read; and if, when the instrument was set up at B, the latter were turned till the needle indicated the same degree, the sides of the table would be parallel to their positions when at A, and the line B*a* would coincide in direction with BA on the ground.

For the use of the plane table in finding, on the plan, the position of the observer's place by means of two or more objects which are visible from thence, and which have been already laid down, see "Treatise on Practical Surveying," Murray, 1829.

OF THE THEODOLITE.

62. The theodolite, or as it is sometimes written, theodolet, is the instrument which is most generally employed for the survey of tracts of land of any extent from that of an estate, or a parish, upwards. It is not intended here to describe the great instruments of this kind (three feet in diameter) which are employed for fixing the principal stations in the trigonometrical survey of a kingdom or empire; and the student is referred to the treatise on Practical Astronomy and Geodesy (Arts. 104. &c.) for a description of the altitude and azimuth circle, which is similar in construction to some of the instruments employed for, at least, the secondary stations. The following account relates merely to those theodolites which are usually employed in the survey of a few square miles of ground when a considerable degree of accuracy is required.



The instrument may be said to consist of two parts, by one of which are measured angles in planes parallel to the horizon, and by the other, angles of elevation or such as are in vertical planes. The whole is supported on a stand with three legs, at the upper end of which is a short pillar of metal terminating below in a ball which rests in a socket: the pillar is made to pass through two circular plates A, B, of brass, of which the lower one, containing the socket just mentioned, is attached to the legs of the stand, and the upper one is capable of being made parallel to the other, or of having to it a small inclination by means of four screws which are equally distant from one another about the pillar: three of these screws appear at *a, a, a*.

Two circular plates, C, D, generally of brass with silver rims, lie one on the other, and from the lower one projects downwards a hollow stem, E, which, when the theodolite is mounted on its stand, is placed over the upper part of the pillar; the latter, having the form of a frustum of a cone, fits accurately the hollow of the stem, and permits the plates C, D, to which it is attached, to turn with it about the axis of the pillar. The upper plate, C, carries the spirit-levels, *c, c*, which are placed at right angles to one another; and when, by means of the four screws, *a, a*, &c. below, the bubble of air in each of the levels is capable of remaining in the middle of the length of the tube containing the spirit, it is evident that the pair of plates C, D is horizontal, or the axis of the pillar on which they turn is in a vertical position.

The rim of the lower circle, D, being graduated, generally in half degrees. and that of the upper one carrying an index with a vernier, the observed angles may be read to minutes of a degree. The upper plate, C, carries a compass box, F; and a magnetised needle traverses on a pivot at its centre, which is vertically over the centre of the circle. If, in addition, the upper plate carried at opposite extremities of a diameter two perforated plates serving as *sights*, like those of the plane table, the instrument would be exactly that which was formerly called a circumferentor, and such an instrument is still used for the same purposes as a theodolite, when a smaller degree of accuracy is considered sufficient.

But a theodolite is furnished with a telescope, GH, resting at *d, d* upon a pair of double inclined planes, forming what is called a Y, the planes being tangents to two collars of bell-metal surrounding the tube at the places of support; each Y is carried on the upper surface of a brass bar or plate, KM, and to the lower surface of the latter is attached a graduated semicircle, NP, of brass, whose curvature is downwards. This semicircle turns on an axle which, when the instrument is adjusted, is in a horizontal position; each of its extremities rests, as at Q, upon a support formed by two brass bars which are inclined to one another like the sides of the letter A, their feet resting on a metal ring surrounding the compass box, or rather on the cylindrical side of the box itself. At three points, equally distant from one another, on the circumference of the ring are capstan-headed screws as *e*, *e*, by which the ring, and consequently the telescope, can have its position altered, so that the latter may be rendered horizontal. Under, or above the telescope is a spirit level, R, which turns on a joint at one extremity, and is provided with a capstan-headed screw at the other for the purposes of adjustment: the spirit levels on the horizontal plate, C, are provided with screws for the like purpose.

In the eye-piece of the telescope are two fine wires which cross each other, generally at right angles; and, when the horizontal angle subtended by the distance between two objects is to be observed, the index of the horizontal circle, C, is made to coincide with the zero of the graduations on D; then, the upper plate being clamped to the lower by the screw at *b*, the whole instrument is to be turned on the vertical axis till the intersection of the wires bisects a mark on one of the two objects, generally that which is on the left of the other. The lower plate being now prevented from turning round by a clamping screw, the end of which appears at *h*, the upper one is released and turned, carrying the telescope with it, till the intersection of the wires is made to bisect a mark on the other object, when the graduation opposite the index will express the angle comprehended between the visual rays from the marks to the centre of the instrument.

This method is not always used: frequently, after having clamped the horizontal circles together, without regarding the zero of the graduations, the instrument is turned round horizontally till the intersection of the wires appears to coincide nearly with one of the objects, when the whole instrument is clamped to its stand by the screw at *h*: the apparent coincidence is then made accurate by means of the tangent screw at *k*, and the graduation opposite the index is read. Next, releasing the upper circle, C, from the lower, the former is turned round, the telescope turning with it, till the intersection of the wires appears nearly to coincide with the other object; the circle, C, is then clamped to D by the screw, *b*, and the apparent coincidence is made accurate by the tangent screw, *m*: the graduation opposite the index is again read; and the difference between the two readings is manifestly the measure of the angle between the objects.

To obtain the angular elevation of an object, the index of the vernier belonging to the vertical circle, N P, must be made to coincide with the zero of the graduations on that circle when the optical axis of the telescope is in a horizontal position; then, raising the object end of the telescope till the intersection of the wires appears to coincide with the mark whose elevation is required, the graduation opposite the index will express that elevation.

Some theodolites have the horizontal axis of motion fixed to the telescope itself, as in a transit instrument; and these are without a circle for ascertaining the altitudes of objects: others are formed with the telescope fixed between two segments of circles, the horizontal axis of motion being above it. Again, some theodolites are provided with an apparatus by which they may be employed as repeating instruments. See "Repeating Circle," in the Treatise on Practical Astronomy and Geodesy, Arts. 126—8.

a. The manner of adjusting a theodolite can scarcely be understood except by performing the operations under the directions of a teacher; and therefore only a brief explanation of the process will be given.

The instrument, mounted on its stand, must be set up on the ground or floor, in such a manner that the spirit levels, *c, c,* on the horizontal plate, C, may be respectively parallel to a line passing through the pillar of the stand and through two of the adjusting screws, *a, a;* and by these screws the bubbles of air must be made to occupy the middles of the tubes containing the spirit. If, then, the bubbles retain those positions during a revolution of the instrument about the vertical axis, it is evident that the plate carrying the levels will be exactly horizontal: this, however, will seldom be the case; therefore, after the adjustment described, turn the instrument half round on its vertical axis, so that each level may be parallel to its former position: then, attending to one level only, should the air bubble move to one end of the tube, cause it to return half way to the middle by means of the capstan-headed screws of the level itself, and the other half by means of the screws, *a, a.* Again, turn the instrument half round, and if, after the motion, the bubble of that level remain in the middle, the level is horizontal, and the plate is also, in that direction, horizontal: but if, after the motion, the bubble has moved towards one end of the tube, the process must be repeated; and, in general, several repetitions are necessary before the level is completely adjusted. Corresponding operations must then be performed for the adjustment of the other level.

In order to adjust the spirit level attached to the telescope, the zero of the graduations on the semicircle, NP, must be made to coincide with the index of the vernier; then, if the bubble of air should not be in the middle of its tube, it must be made to occupy that place by the three capstan-headed screws, *e, e, e.* The telescope must, next, be taken out of the supports, *d, d,* and replaced in a reversed position (the object-glass being brought where the eye-glass was before); and if the bubble should settle in the middle of the tube, the adjustment is made: if otherwise, the bubble moving towards one end of its tube, it must be brought to the middle, half by means of the capstan-headed screws, *e, e,* and the other half by means of the like screw at one end of the level itself. After several repetitions of these operations, the bubble of air may be made to retain its place in the middle, while the instrument is turned quite round horizontally: thus the plates by which the horizontal angles are to be obtained are parallel to the horizon, and the telescope is also in a horizontal position when the index of the vernier is at the zero of the graduations on the vertical arc.

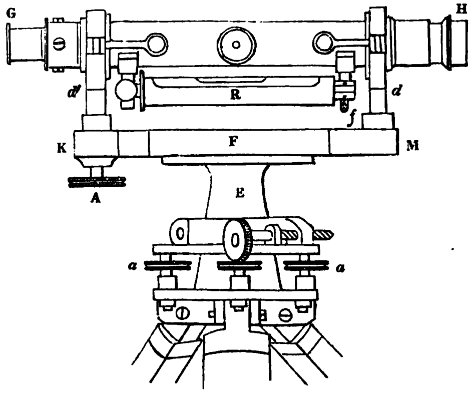
There is still required the adjustment of what is called the line of collimation, that is, the process of making the intersection of the wires in the telescope coincide with the optical axis of the latter; and the operations for this purpose should be performed previously to making the adjustments before mentioned. By means of the pinion, s, at the top or in the side of the telescope, make the foci of the object and eye-glasses coincide with one another; which may be ascertained by distinct vision being obtained of the wires and of some object at any given distance, and, at the same time, the intersection of the wires appearing not to change its place on the object when the eye is moved a little way about the centre of the aperture in the eye-piece of the telescope: then, having clamped the vertical circle to its frame by means of the screw, *n,* and having turned the telescope on its axis till one of the wires within it is parallel to the horizon, by means of the tangent screw, *p,* cause the horizontal wire to appear in coincidence with a well-defined object at a distance. Now, turning the telescope half round on its axis, when the same wire will be again in a horizontal position; if in this state it coincide with the same object, that wire passes through the optical axis of the telescope; if not, it must, by means of the two screws, *q, r,* be moved parallel to itself half way to the object. Again, by means of the screw, *p,* make the wire coincide with the object, and turn the telescope half round on its axis; then, if the coincidence exist, the adjustment is made: otherwise, the former process must be repeated till an apparent coincidence of the wire with the object takes place in both positions. The other wire must be adjusted in like manner.

OF THE SPIRIT LEVEL.

63. For many of the purposes to which a spirit-level is applied, a moderately-good theodolite will suffice, but when the relative heights of points on the ground are required with considerable accuracy, a telescope provided with a spirit-tube of considerable length should be employed; therefore, instruments for the express purpose of levelling, as the operation of determining the relative heights is called, are constructed: these are designated spirit-levels.

In general, the stand, which has at the top a short pillar passing through two plates and terminating above in the form of a frustum of a cone, is similar to that which supports a theodolite. The telescopes and spirit-tubes of both kinds of instrument are also, in general, similar, and are disposed in like manner.

The telescope, GH, is supported, near its extremities, between double inclined planes or Y's at *d*, *d′*, above a horizontal bar, KM, at the middle of which is a compass-box, F, in which traverses a magnetised needle. The lower extremity of *d* is attached to the horizontal bar, and is provided with a joint by which, the telescope being in its place, it is capable of turning a little way in a vertical plane. The stem, *d′*, of the other Y passes quite through the bar, and in its lower part is formed a concave screw: a convex screw, whose head is A, turns in the other; and thus, for the purposes of adjustment, the stem, *d′*, and with it the end, G, of the telescope may be raised or lowered. Below the compass-box is a socket of brass, similar to that of a theodolite, which, when the instrument is placed on its stand, receives the conical top of the



pillar before mentioned; and upon this the whole instrument is capable of turning round horizontally.

The adjustment of the line of collimation is similar to that of the corresponding line in the telescope of the theodolite, while the spirit-tube, R, is rendered parallel to the axis of the telescope, partly by means of the capstan-headed screw, *f*, at one end of the level, and partly by means of the screw, A, which moves the telescope up and down. After these operations have been performed, the bubble of air is made to occupy the middle of the spirit-tube, during a revolution of the instrument upon the pillar at the top of the stand, partly by means of the four screws, *a*, *a*, &c., about that pillar, and partly by means of the screw at A before mentioned.

The levelling staff usually consists of two parts whose transverse sections are rectangular; and one of these, having a dove-tailed groove along it, is enabled to slide upon the other, which is furnished with a corresponding tongue: each part being 6 feet long, the whole length of the rod may, when required, be nearly 12 feet; and along the face and back of one of the parts is a scale of feet and inches, these last being generally divided into tenths.

When the difference between the heights of the ground at the places where the instrument and the staff are set up does not exceed four feet, a transverse vane of wood is made to slide up or down the staff, the latter being held in a vertical position with its lower extremity resting on the ground, or rather upon a picket driven into the ground, till its head is on a level with the surface: the vane is made fast, by the assistant who holds the staff, at the place where its upper edge appears to a person looking through the telescope of the spirit-level (the latter having been previously adjusted) to be in contact with the horizontal wire in the eye-piece. The height of the vane above the ground at the foot of the staff expresses the depression of that spot below a horizontal plane passing through the optical axis of the telescope.

When the difference between the heights of the ground is greater than 4 feet and less than 7 feet 6 inches, the vane is fixed transversely at 3 inches below the top of one part of the rod; and, the lower extremity of the other part resting on the ground, the former part is raised up till, on looking through the telescope, the upper edge of the vane appears to be in coincidence with the horizontal wire: then, by means of the scale at the back of the sliding-rod, the height of the vane above the ground is ascertained.

With a staff graduated in the usual way, the assistant who holds it must read on its scale the height of the vane from the ground; but Gravatt's rod, which is divided into hundredths of a foot, has the spaces between the lines of division alternately black and white, so that the person who looks through the telescope is enabled to read the heights: this evidently saves him some trouble, and renders him, in respect of the reading, independent of the assistant.

Forms different from those which have been described have been given both to theodolites and spirit-levels. (See "A Treatise on Mathematical Instruments, &c.," by F. W. Simms, Weale, 1838.)

OF THE POCKET SEXTANT.

64. The pocket sextant which, on account of its portability, is frequently employed to measure the angles between the visual rays from any terrestrial objects to the eye of the observer is, in principle, the same as the sextant used for nautical observations, and described in the Treatise on Practical Astronomy and Geodesy (Arts. 129—132): it is frequently also provided with a divided index or horizon-glass, agreeably to the constructions proposed by Captains Fitzroy and Beechey (*Ib.* Arts. 138, 139.), for the purpose of observing angles greater than 120°. That the instrument may not be injured in being carried, it is contained within a cylindrical box, and it is generally now provided with a small telescope. By means of the vernier an observed angle may be read on the arc to minutes of a degree; but the instrument is subject to a certain disadvantage, in the pinion by which the index is turned being unavoidably very near the centre of motion.

The manner of using the instrument is well known; it may, therefore, be sufficient to say that its face must be as nearly as possible in a plane passing through the eye and the two objects between which the angle is to be taken; and, when held uppermost, the observer must look through the unquicksilvered part of the horizon-glass at the left hand object; then, turning the pinion by which the index and its mirror are made to revolve, the reflected image of the right hand object must be made to coincide with the other at the line separating the quicksilvered from the unquicksilvered part of the glass, when the value of the subtended angle may be read on the graduated arc. If the sextant is held with its face downwards, the reflection of the left hand object must be made to coincide in like manner with the right hand object seen by direct vision in the horizon-glass.

When the sextant is employed for the purpose of fixing the relative positions of any three stations, in the manner of a trigonometrical survey, it is obvious that the objects which serve as stations should be as nearly as possible in a horizontal plane; and the stations should be, at least, half a mile from each other, in order that the parallax of the instrument may be insensible.

The sextant is occasionally employed to obtain the angular elevations of objects above the horizon; in which case the angle must be taken between the object and its image, seen by reflection in an artificial horizon. The angle thus taken is double the required altitude (Astron. and Geod. Art. 133.), the error arising from the distance of the observer's eye from the reflecting surface being neglected.

The artificial horizon usually employed is a shallow cylindrical or rectangular vessel of wood, earthenware, or metal, containing mercury from the surface of which the light coming from the object is reflected to the eye of the observer. In order to prevent the surface of the fluid from being agitated by the air, a circular plate of glass having its surfaces parallel to each other is laid upon it when it is contained in a cylindrical vessel: a rectangular trough is usually covered by a frame containing two plates of glass which form inclined planes meeting in a ridge over the middle of the trough.

Since, on a journey, it is not always convenient to carry mercury, there may be used an artificial horizon consisting of a circular mirror of dark glass or of polished metal, fixed in a frame about which at equal distances are placed three adjusting screws. By these, a small spirit-level being laid on the mirror, and the latter being placed on a stone or other object which may serve as a pedestal, the reflecting surface may be made parallel to the natural horizon.

a. When an artificial horizon is not at hand, or the low elevation of the object prevents the reflected image of the latter from being seen in such horizon, the angular altitude may be obtained roughly by means of the simple instrument called a Gunter's quadrant. This is the fourth part of a circle made of box-wood having its arc graduated in degrees: a small plummet is suspended from the angle of the quadrant, and two perforated sight-vanes are placed on one of its straight edges. The instrument being held in a vertical position with one extremity of the arc near the eye, the line of the *sights* is directed to the elevated object, and the plummet is allowed to hang freely: then the degrees on the arc between the plumb-line and the extremity which is furthest from the eye will express the required angle of elevation.

OF THE REFLECTING SEMICIRCLE.

65. This instrument, which was invented by Lieutenant-General Sir Howard Douglas, being frequently employed instead of a pocket sextant for the purpose of fixing the positions of stations, a brief description of it may with propriety be introduced here.

The instrument is of brass, and the index-glass, which makes a constant angle with a plane passing through its middle point and the centre of the semicircle, perpendicularly to the plane of the latter, is fixed, also perpendicularly to that plane, to a bar which turns upon the centre: the horizon-glass is attached in like manner to one end of a bar which turns on a pivot at that extremity of the instrument which is opposite the index-glass. The other extremity of this bar carries the sight-vane, and the bar itself being perforated in the direction of its length, on being turned upon the axis of the horizon-glass, a pin rising through the perforation from a plate forming part of the bar carrying the index-glass causes both glasses to turn at the same time, one upon a point in the circumference and the other upon the centre of the semicircle. By this construction the angle described by the index-bar from its position, when the mirrors were parallel to one another, is always equal to twice the angle at which the mirrors are inclined to one another; but the angle subtended by the distance between two objects, placed in apparent contact after two reflections, is equal to twice the inclination of the mirrors (Astron. and Geod. art. 132.) Consequently, the subtended angle is equal to that which is described by the index-bar, and its value is given by the graduations on the semicircle. One side of this bar having a chamfered edge in the direction of a radius of the semicircle, it is evident that after an angle has been observed, it may be immediately protracted on paper by means of the instrument itself.

OF THE VERNIER.

66. The vernier, so called from the name of its inventor, is a scale of small extent graduated in such a manner that, being moved by the side of a fixed scale or being fixed while this scale is moved, by the coincidence of one of its lines of division with one of those on the scale to which it is applied, there may be obtained the value of a space on the latter expressed in parts of its divisions.

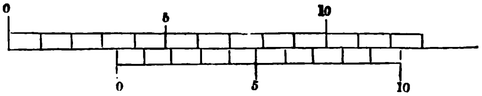
If each division of any scale be considered as a unit, and it be required, by means of the vernier, to obtain a space equal to any portion, as $\frac{1}{n}$th of that unit, the extent of the scale called the vernier must be made equal to that of $n-1$ of those divisions, and this extent on the vernier scale must be divided into n equal parts: it is manifest, then, that each division of the vernier will be equal to a portion of the unit division on the instrument expressed by $\frac{n-1}{n}$; and subtracting this from 1, we have $\frac{1}{n}$ for the fraction of the unit division by which the latter exceeds the extent of one division on the vernier-scale. Thus, if it be required to obtain, by means of the vernier, subdivisions of any scale equal to one-tenth of each division on that scale, the vernier-scale must be equal in extent to nine of these divisions, and must be divided into ten equal parts: then the excess of each division on the first scale above each of those on the vernier will be equal to one-tenth of the former, as required.

Each circle of a theodolite is usually divided into half degrees, and by means of the vernier these are to be subdivided into minutes or into spaces equal to $\frac{1}{30}$th of half a degree. The extent of the vernier-scale is, therefore, made equal to 29 half degrees, or 870 minutes, and this is divided into 30 equal parts: each part is, therefore, equal to 29 minutes, and the excess of each division on the circle above each division of the vernier is equal to one minute.

Again, the arcs of sextants are divided into degrees which, in some cases, are subdivided into four equal parts, each of which is consequently equal to 15 minutes; and, on instruments so divided it is customary to obtain, by means of the vernier, subdivisions each of which is equal to 15 seconds, or to $\frac{1}{60}$th of each division on the arc. For this purpose the extent of the vernier-scale is made equal to 59 divisions of the arc ($= 14^\circ\ 45'$ or $885'$), and is divided into 60 equal parts; each of these parts is consequently equal to $14'\ 45''$, and the excess of each division on the arc above each division of the vernier is 15 seconds.

In order, now, to read the value of the space between the index of the vernier, when it stands between two lines of division, and the first of these two lines, it is merely necessary to look along the two scales till some division line of the vernier is found to be in coincidence (or very nearly) with a division of the first scale: the number of that line of division on the vernier, reckoning from the index line which is considered as that of zero, will express the required value of the space in subdivisions of the scale.

For example, if the divisions of a scale are to be subdivided by the vernier into tenths, and the index of the latter stand



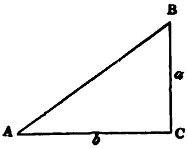
between the third and fourth division of the scale; on looking along the vernier, it may be found that the fourth line of division is in coincidence with one of the division lines of the scale on the instrument: it will follow that the index line 0 of the vernier has passed over four-tenths of a division beyond the third on the scale, and thus the whole space passed over from the zero of the scale is 3.4 divisions. It is easy to perceive, from this example, how the value of an observed angle is to be read in degrees and minutes, or in degrees, minutes, and fractions of a minute, on the circle of a theodolite or on the arch of a sextant.

SECTION VII.

MENSURATION OF HEIGHTS AND DISTANCES.

67. To find the height of an accessible object on a horizontal plane.

Let BC be the height which is to be determined.



In the horizontal plane passing through C, the foot of the object, let a base CA $= b$, be measured (Art. 60.), and at the extremity A of the base, let the angle of elevation BAC = A, be measured. (Art. 62.)

Then, in the right-angled triangle ABC, A, b, are given to find a;

$$\therefore\ a = b \text{ tan. A.} \quad \text{(Art. 56.)}$$

$$\text{and log. } a = \text{log. } b + \text{log. tan. A} - 10.$$

Ex. Given $b = 150$ feet; $\text{A} = 35^\circ\ 41'$: required a.
Ans. $a = 107.7195$.

a. If the angles of elevation are measured with a theodolite, it will be necessary to add the height of the instrument (commonly 5 feet) to the heights obtained by calculation.

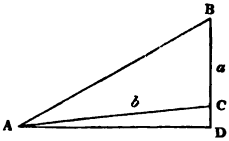
Adding 5 feet to the value of a in the preceding example,

$$\text{BC or } a = 107.7195 + 5 = 112.7195 \text{ feet.}$$

In the examples contained in this section, all the calculated heights are augmented by 5 feet.

68. To find the height of an accessible object on sloping ground.

Let BC be the height which is to be determined.



In the inclined plane passing through C, let a base CA $= b$, be measured (Art. 60.); and at the extremity A of the base, let the angle of elevation BAD = A, and the angle of inclination of the sloping base, CAD = A′, be measured. (Art. 62.)

Then A, A′, b, and the right angle D, are given to find BC or a.

$<\mathrm{BAC} = \mathrm{BAD} - \mathrm{CAD} = \mathrm{A} - \mathrm{A}'$. $\mathrm{B} = 90° - \mathrm{A}$ $\therefore \sin. \mathrm{B} = \cos. \mathrm{A}$ (Art. 17.)

Wherefore, since $\mathrm{BC} = \dfrac{\mathrm{AC} \times \sin. \mathrm{BAC}}{\sin. \mathrm{B}}$. (Art. 59. Case I.)

$$a = \frac{b \sin. (\mathrm{A} - \mathrm{A}')}{\sin. \mathrm{B}} = \frac{b \sin. (\mathrm{A} - \mathrm{A}')}{\cos. \mathrm{A}},$$

And, $\log. a = \log. b + \log. \sin. (\mathrm{A} - \mathrm{A}') +$ arith. comp. of $\log. \cos. \mathrm{A} - 10$.

Ex. Given $b = 275$ feet; $\mathrm{A} = 29° \, 35'$; $\mathrm{A}' = 6° \, 14'$: required a.

Ans. 130.334 feet.

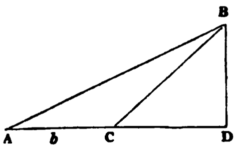
a. If BD, the height of B above the horizontal plane passing through A, and AD the horizontal distance of A from BD are required.

1st. $\mathrm{AD} = b \cos. \mathrm{A}'$ (Art. 58. Case I.)
2d. $\mathrm{BD} = \mathrm{AD} \tan. \mathrm{A}$ (Art. 58. Case II.) $= b \cos. \mathrm{A}' \tan. \mathrm{A}$.

69. To find the height and distance of an inaccessible object on a horizontal plane.

Let BD be the height, and AD the distance, which are to be found.

1st. Let the base $\mathrm{AC} = b$, be measured from the point A, in the direction passing through D, the foot of the inaccessible object; and at A, C, the extremities of the base, let the angles of elevation, $\mathrm{BAC} = \mathrm{A}$, $\mathrm{BCD} = \mathrm{C}$, be measured. The base b, with the angles of elevation A, C, and the right angle D, are sufficient data for the determination of BD, AD.



In the triangle ABC,

$$<\mathrm{ABC} = \mathrm{BCD} - \mathrm{BAC} \text{ (Euc. I. 32.) } = \mathrm{C} - \mathrm{A};$$
and, $\sin. \mathrm{ACB} = \sin. \mathrm{BCD}$ (Art. 18. *c.*) $= \sin. \mathrm{C}$.

Also, $\mathrm{AB} = \dfrac{\mathrm{AC} \sin. \mathrm{ACB}}{\sin. \mathrm{ABC}}$. (Art. 59. Case I.)

$$\text{or, } \mathrm{AB} = \frac{b \sin. \mathrm{C}}{\sin. (\mathrm{C} - \mathrm{A})}.$$

Again, in the triangle ABD,

$$\mathrm{BD} = \mathrm{AB} \sin. \mathrm{BAC} \text{ (Art. 58. Case I.) } = \frac{b \sin. \mathrm{A} \sin. \mathrm{C}}{\sin. (\mathrm{C} - \mathrm{A})}.$$

$$\mathrm{AD} = \mathrm{AB} \cos. \mathrm{BAC} = \frac{b \cos. \mathrm{A} \sin. \mathrm{C}}{\sin. (\mathrm{C} - \mathrm{A})}.$$

If CD is required; $\mathrm{CD} = \mathrm{AD} - \mathrm{AC}$.

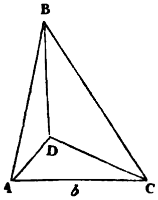
Hence,

log. BD = log. b + log. sin. A + log. sin. C + arith. comp. of log. sin. (C − A) − 20.
log. AD = log. b + log. cos. A + log. sin. C + arith. comp. of log. sin. (C − A) − 20.

Ex. Given $b = 225$ feet; $\mathrm{A} = 24° \; 38'$; $\mathrm{C} = 39° \; 27'$: required BD, AD?

Ans. BD = 238.02 feet; AD = 508.18 feet.

2d. If a base cannot be measured from the point A, in the direction passing through D, the foot of the inaccessible object, let a base AC = b, be measured in any convenient direction.



At the extremity A of the base, let the horizontal angle CAD = A, and the angle of elevation BAD = A′, be measured; and at the other extremity, C, of the base, let the horizontal angle ACD = C be measured. The base b, with the horizontal angles A, C, the angle of elevation A′, and the right angle ADB, are sufficient data for the determination of AD, CD, the horizontal distances of the ends of the base from the foot of the object, and of BD, the height of the object above the plane ACD.

In the triangle ACD, $\angle \mathrm{ADC} = 180° - (\mathrm{A} + \mathrm{C})$. (Euc. I. 32.)

$$\therefore \text{ sin. ADC} = \text{sin. } (\mathrm{A} + \mathrm{C}). \quad (\text{Art. 18. } c.)$$

$$\mathrm{AD} = \frac{\text{AC sin. ACD}}{\text{sin. ADC}} = \frac{b \text{ sin. C}}{\text{sin. } (\mathrm{A} + \mathrm{C})}. \quad (\text{Art. 59. Case I.})$$

$$\mathrm{CD} = \frac{\text{AC sin. CAD}}{\text{sin. ADC}} = \frac{b \text{ sin. A}}{\text{sin. } (\mathrm{A} + \mathrm{C})}.$$

And in the triangle ABD,

$$\mathrm{BD} = \text{AD tan. BAD (Art. 58. Case II.)} = \frac{b \text{ sin. C tan. A}'}{\text{sin. } (\mathrm{A} + \mathrm{C})}.$$

∴ log. AD = log. b + log. sin. C + arith. comp. of log. sin. (A + C) − 10.
log. CD = log. b + log. sin. A + arith. comp. of log. sin. (A + C) − 10.
log. BD = log. b + log. sin. C + log. tan. A′ + arith. comp. of log. sin. (A + C) − 20.

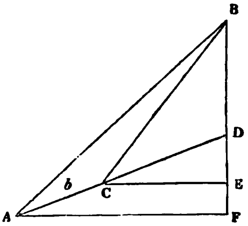
Ex. Given $b = 486$ feet; $\mathrm{A} = 56° \; 48'$; $\mathrm{A}' = 19° \; 54'$; $\mathrm{C} = 42° \; 17'$: required AD, CD, BD.

Answer. AD = 331.132 feet; CD = 411.8326 feet; BD = 124.868 feet.

a. The horizontal distances and the height of the object may also be obtained by computation, when the oblique angles BAC, BCA, have been observed with a sextant (Art. 64.), and the angle of elevation BAD, either with a sextant and an artificial horizon, or (when great accuracy is not required) with the simple instrument called a Gunter's quadrant. (Art. 64. *a.*)

70. To find the height and horizontal distance of an inaccessible object on sloping ground.

Let BD (or BF) be the height, and AF the distance, which are to be found.



1st. Let the base AC $= b$, be measured from the point A in the direction passing through D, the foot of the inaccessible object: at A let the angle of elevation BAF $=$ A, and the angle of inclination of the slope DAF $=$ A′, be measured; also at C, let the angle of elevation BCE $=$ C be measured. Then b, A, A′, C, and the right angle F, are given to find BD, BF, and AF.

$\angle \mathrm{DCE} = \mathrm{DAF} = \mathrm{A}'$, (Euc. I. 29.)

$\angle \mathrm{BCD} = \mathrm{BCE} - \mathrm{DCE} = \mathrm{C} - \mathrm{A}'$: $\angle \mathrm{BAC} = \mathrm{BAF} - \mathrm{DAF} = \mathrm{A} - \mathrm{A}'$

$\angle \mathrm{ABC} = \mathrm{BCD} - \mathrm{BAC}$, (Euc. I. 32.) $= \mathrm{C} - \mathrm{A}$.

Sin. ACB $=$ sin. BCD $=$ sin. (C $-$ A′)
Sin. ADB $=$ sin. ADF $=$ cos. DAF $=$ cos. A′

In the triangle ABC, $\mathrm{AB} = \dfrac{\mathrm{AC}\ \sin.\ \mathrm{ACB}}{\sin.\ \mathrm{ABC}} = \dfrac{b \sin.\ (\mathrm{C}-\mathrm{A}')}{\sin.\ (\mathrm{C}-\mathrm{A})}$

(Art. 59. Case I.)

In the triangle ABD,

$$\mathrm{BD} = \frac{\mathrm{AB}\ \sin.\ \mathrm{BAD}}{\sin.\ \mathrm{ADB}} = \frac{b \sin.\ (\mathrm{C}-\mathrm{A}')\ \sin.\ (\mathrm{A}-\mathrm{A}')}{\sin.\ (\mathrm{C}-\mathrm{A})\ \cos.\ \mathrm{A}'}$$

In the triangle ABF, $\mathrm{BF} = \mathrm{AB} \sin.\ \mathrm{BAF} = \dfrac{b \sin.\ (\mathrm{C}-\mathrm{A}')\ \sin.\ \mathrm{A}}{\sin.\ (\mathrm{C}-\mathrm{A})}$

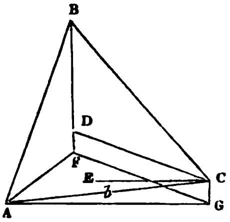
$$\mathrm{AF} = \mathrm{AB} \cos.\ \mathrm{BAF} = \frac{b \sin.\ (\mathrm{C}-\mathrm{A}')\ \cos.\ \mathrm{A}}{\sin.\ (\mathrm{C}-\mathrm{A})}$$

∴ log. BD $=$ log. b + log. sin. (C $-$ A′) + log. sin. (A $-$ A′) + arith. comp. of log. sin. (C $-$ A) + arith. comp. of log. cos. A′ $-$ 20.
log. BF $=$ log. b + log. sin. (C $-$ A′) + log. sin. A + arith. comp. of log. sin. (C $-$ A) $-$ 20.
log. AF $=$ log. b + log. sin. (C $-$ A′) + log. cos. A + arith. comp. of log. sin. (C $-$ A) $-$ 20.

Ex. Given $b = 500$ ft.; A $= 32° 26′$; A′ $= 4° 42′$; C $= 40° 5′$: required BD, BF, AF.

Ans. BD $=$ 1020.5 ft.; BF $=$ 1171.42 ft.; AF $=$ 1835.628 ft.

2d. If a base cannot be measured from A, in the direction passing through the foot of the inaccessible object; let a sloping base AC $= b$ be measured in any convenient direction.



Let it be supposed that from the foot of the inaccessible object and the upper end of the base, perpendiculars are let fall upon the horizontal plane passing through A, the lower end of the base, (Planes. Prop. 10.) and that these perpendiculars meet this plane in F, G, respectively. Let AG be the intersection of the vertical plane ACG with the horizontal plane passing through A; and let AF, GF be joined. Through C, in the plane ACG, let CE be drawn parallel to AG; and, in the plane BFG, CD, parallel to FG; and let AB be joined.

Then CG=DF; CD=GF, (Euc. I. 34.) and <DCE=AGF, (Planes, Prop. 9.). Also, GAF is the horizontal angle made at A by the direction of the base, and the line of sight passing through A and the inaccessible object; DCE=FGA is the horizontal angle made at C by the direction of the base and the line of sight passing through C and the inaccessible object; CAG is the angle of inclination of the base AC; and BAF is the angle of elevation of the top of the inaccessible object. Let these angles be measured with a theodolite, (Art. 62.) and let GAF = A, DCE or FGA = C, CAG = A′, FAB = A″.

In the triangle ACG, $\mathrm{AG}=\mathrm{AC}\cos.\mathrm{GAC}=b\cos.\mathrm{A}'$.

In the triangle AFG,

$$\sin.\mathrm{AFG}=\sin(\mathrm{GAF}+\mathrm{FGA})=\sin.(\mathrm{A}+\mathrm{C}),\ (\text{Art. 19. } b.)$$

$$\mathrm{AF}=\frac{\mathrm{AG}\sin.\mathrm{FGA}}{\sin.\mathrm{AFG}}=\frac{b\cos.\mathrm{A}'\sin.\mathrm{C}}{\sin.(\mathrm{A}+\mathrm{C})}$$

$$\mathrm{GF}=\frac{\mathrm{AG}\sin.\mathrm{GAF}}{\sin.\mathrm{AFG}}=\frac{b\cos.\mathrm{A}'\sin.\mathrm{A}}{\sin.(\mathrm{A}+\mathrm{C})}$$

And in the triangle ABF,

$$\mathrm{BF}=\mathrm{AF}\tan.\mathrm{FAB}=\frac{b\cos.\mathrm{A}'\sin.\mathrm{C}\tan.\mathrm{A}''}{\sin.(\mathrm{A}+\mathrm{C})}$$

$\therefore$ log. AF = log. b + log. cos. A′ + log. sin. C + arith. comp. of log. sin. (A + C) − 20.
log. GF = log. b + log. cos. A′ + log. sin. A + arith. comp. of log. sin. (A + C) − 20.
log. BF = log. b + log. cos. A′ + log. sin. C + log. tan. A″ + arith. comp. of log. sin. (A + C) − 30.

Ex. Given $b = 477.6$ ft.; A = 74° 18′; C = 48° 52′; A′ = 9° 26′; A″ = 32° 29′: required AF, GF, BF.

Ans. AF = 423.918 ft.; GF = 541.839 ft.; BF = 274.892 ft.

a. If BD, the height of B above C, the upper end of the base, is required:

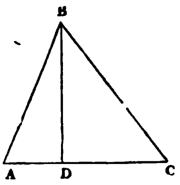
$$\mathrm{BD}=\mathrm{BF}-\mathrm{DF}=\mathrm{BF}-\mathrm{CG}=\mathrm{BF}-b\sin.\mathrm{A}'.$$

Otherwise, let the angle of elevation BCD ($=$ C′) be measured:

$$\text{Then, BD} = \text{CD} \times \tan.\ \text{C}' = \text{GF} \times \tan.\ \text{C}' = \frac{b \cos.\ \text{A}' \sin.\ \text{A} \tan.\ \text{C}'}{\sin.(\text{A}+\text{C})}.$$

b. The horizontal distances and the height of the object may also be obtained by computation when the oblique angles BAC, BCA, have been observed with a sextant (Art. 64.), and the angles of elevation with a Gunter's quadrant (Art. 64.*a*.) or otherwise.

71. Given the distance between two stations, and the horizontal angle contained, at each station by two lines of sight, of which the one passes through the other station and the other through an inaccessible object; to find the distance of the inaccessible object from each station, and also from the straight line joining the two stations.



Let the stations be A, C, and the inaccessible object B: then the angles A, C, the base AC$=b$, and the right angle at D, are given to find AB, CB, DB.

In the triangle ABC, sin. ABC $=$ sin. (A$+$C)

$$\text{AB} = \frac{\text{AC} \sin.\ \text{ACB}}{\sin.\ \text{ABC}} = \frac{b \sin.\ \text{C}}{\sin.\ (\text{A}+\text{C})}$$

$$\text{CB} = \frac{\text{AC} \sin.\ \text{BAC}}{\sin.\ \text{ABC}} = \frac{b \sin.\ \text{A}}{\sin.\ (\text{A}+\text{C})}$$

$$\text{DB} = \text{AB} \sin.\ \text{BAD} = \frac{b \sin.\ \text{C} \sin.\ \text{A}}{\sin.\ (\text{A}+\text{C})}.$$

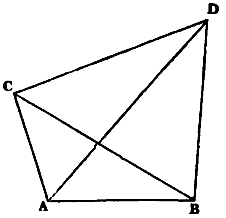
$\therefore$ log. AB $=$ log. b $+$ log. sin. C $+$ arith. comp. of log. sin. (A $+$ C) $-$ 10.
log. CB $=$ log. b $+$ log. sin. A $+$ arith. comp. of log. sin. (A $+$ C) $-$ 10.
log. DB $=$ log. b $+$ log. sin. C $+$ log. sin. A $+$ arith. comp. log. sin. (A$+$C) $-$ 20.

Ex. Given $b=1328$ ft.; A$=84°\ 23'$; C$=43°\ 19'$; ADB $=$ 90°: required AB, CB, DB.

Ans. AB $=$ 1151.44 ft.; CB $=$ 1670.3505 ft.; DB $=$ 1145.913 ft.

72. To find the distance between two inaccessible objects which can be seen from two given stations in the same plane.

Let C, D, be the inaccessible objects whose horizontal distance is required.



Assume two stations A, B, in a horizontal plane: measure the base AD $=b$; and the horizontal angles BAD $=$A, DAC$=$A′; ABC$=$B; CBD$=$B′.

Then, in the triangle ACB,

$$\angle \text{ACB} = 180° - (\text{A}+\text{A}'+\text{B})$$

$$\therefore \sin.\ \text{ACB} = \sin.\ (\text{A}+\text{A}'+\text{B})$$

(Art. 19. *b*.)

$$\text{And AC} = \frac{\text{AB} \sin.\ \text{ABC}}{\sin.\ \text{ACB}} = \frac{b \sin.\ \text{B}}{\sin.\ (\text{A}+\text{A}'+\text{B})}.$$

Again, in the triangle ADB, $\angle$ ADB $= 180° - (A + B + B')$,

$$\therefore \ \sin. ADB = \sin. (A + B + B'),$$

$$\text{and } AD = \frac{AB \sin. ABD}{\sin. ADB} = \frac{b \sin. (B + B')}{\sin. (A + B + B')}.$$

Lastly, in the triangle ACD,

$$CD = \sqrt{\{AC^2 + AD^2 - 2\,AC \times AD \cos. A'\}} \text{ (Art. 57. } c.\text{)},$$

$$\text{or, } CD = \sqrt{\left\{\frac{b^2 \sin.^2 B}{\sin.^2(A+A'+B)} + \frac{b^2 \sin.^2(B+B')}{\sin.^2(A+B+B')} - 2 \times \frac{b \sin. B}{\sin. (A+A'+B)} \times \frac{b \sin. (B+B')}{\sin.(A+B+B')} \times \cos. A'\right\}}$$

In computing CD from this formula, by logarithms, it will be necessary to calculate separately the value of each of the three terms in the second member. As nearly the same quantities enter into each term, the labour is not great.

Otherwise, after AC and AD have been determined, since the angle CAD is known, the value of either of the angles ACD, ADC, may be found by Art. 59. Case III., and the value of CD by Art. 59. Case I.

Ex. Given $b = 880$ yards, $A = 48° 54'$; $A' = 52° 46'$; $B = 32° 29'$; $B' = 63° 15'$: required CD. *Ans.* CD $= 1231.4$ yards.

$$\text{Calculation of } AC^2 = \frac{b^2 \sin.^2 B}{\sin.^2(A + A' + B)}.$$

$$A + A' + B = 48° 54' + 52° 46' + 32° 29' = 134° 9'.$$

$$\begin{array}{l} \text{Log.}(AC^2) = \\ 2 \log. AC = \end{array} \left\{ \begin{array}{lr} \log. (b^2) = 2 \log. b = 2 \log. 880 \quad . \quad . \quad . & 5.888966 \\ + \log. (\sin.^2 B) = 2 \log. \sin. B = 2 \log. \sin. 32° 29' & + 19.460036 \\ \left. \begin{array}{l} + \text{arith. comp. } \log. \sin.^2 (A + A' + B) = 2 \times \\ \quad \text{arith. comp. of } \log. \sin. (A + A' + B) = \\ \quad 2 \text{ arith. comp. } \log. \sin. 134° 9' \quad . \quad . \end{array} \right\} & + 80.288334 \\ -100 \ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & -100.000000 \end{array} \right.$$

$$\text{The number of this log. is } 433846.5 \quad . \quad . \quad . \quad . \quad 5.637336$$

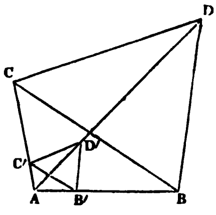
$$\therefore AC^2 = 433846.5.$$

$$\text{Similarly, } AD^2 = \frac{b^2 \sin.^2 (B + B')}{\sin.^2 (A + B + B')} = 2288453.2.$$

Also,

$$2\,AC \times AD \cos. A' = 2 \times \frac{b \sin. B}{\sin. (A + A' + B)} \times \frac{b \sin. (B + B')}{\sin. (A + B + B')} \times \cos. A' = 1205783.3.$$

$$\therefore CD = \sqrt{\{433846.5 + 2288453.2 - 1205783.3\}} = \sqrt{1516516.4} = 1231.4 \text{ yards.}$$



a. Conversely, if the distance CD, and the angles BAD $=$ A; DAC $=$ A'; ABC $=$ B; CBD $=$ B', are given to find AB.

In the direction AB, assume the base AB' $= b$; and at the point B' make the angles AB'C' $=$ ABC; AB'D' $=$ ABD, and join C'D'.

Then, since B′C′ is parallel to BC (Euc. I. 28.); AB′ : AB :: AC′ : AC (Euc. VI. 2.)
And, since B′D′ is parallel to BD; AB′ : AB :: AD′ : AD
Whence, by equality of ratios; AC′ : AC :: AD′ : AD.

Therefore, C′D′ is parallel to CD. (Euc. VI. 2.)

Next, AB′ and the angles B′AD′; D′AC′, AB′C′, C′B′D′, being known, C′D′ is found by the first part of this article.

Lastly, C′D′, CD and AB′ being known, AB is found by the proportion

$$C'D' : CD \; (:: AD' : AD) :: AB' : AB.$$

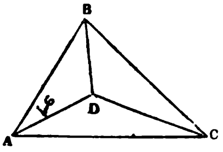
Ex. Given $CD = 1562.8$ yards; $A = 37° 40′$; $A′ = 72° 20′$; $B = 38° 20′$; $B′ = 79° 10′$: required the distance AB.
Ans. $AB = 750$ yards.

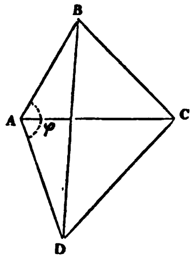
For a geometrical construction of this problem, see Practical Astronomy and Geodesy, Art. 424.

73. To determine the position of an object when there have been observed the angles contained between lines imagined to be drawn from it to three stations whose distances from each other have been previously determined.

Let the distances $BC = a$; $AC = b$; $AB = c$; and the angles $ADB = D$; $BDC = D'$, be given to find the distance of the point D from each of the stations A, B, and C.

Since in the triangle ABC, the three sides a, b, c, are given, the angle ABC, or B, is given. (Art. 59. Case IV.)





And since in the quadrilateral figure ABCD, the angles B and $D + D'$ are known, if the angle BAD were known, then BCD ($= 4$ right angles $- B - D - D' - BAD$, Euc. I. 32. Cor.) would be known. Consequently in each of the triangles ABD, CBD, there would be given one side and two angles to find the sides DA, DB, DC.

Let therefore the angle $BAD = \phi$: then $BCD = 360° - (B + D + D' + \phi)$;
or, making $360° - (B + D + D') = a$; $BCD = a - \phi$.

Now, in the triangle ABD,

$$BD = \frac{AB \sin. BAD}{\sin. ADB} = \frac{c \sin. \phi}{\sin. D} \quad . \; . \; . \quad 1$$

And in the triangle BCD,

$$\text{BD} = \frac{\text{BC sin. BCD}}{\text{sin. BDC}} = \frac{a \text{ sin. } (\alpha - \phi)}{\text{sin. D}'} \quad . \quad . \quad . \quad 2$$

$$\therefore \text{ from equations 1 and 2, } \frac{c \text{ sin. } \phi}{\text{sin. D}} = \frac{a \text{ sin. } (\alpha - \phi)}{\text{sin. D}'},$$

$$\therefore\ c \text{ sin. D}' \text{ sin. } \phi = a \text{ sin. D sin. } (\alpha - \phi),$$

or, since $\text{sin. } (\alpha - \phi) = \text{sin. } \alpha \text{ cos. } \phi - \text{cos. } \alpha \text{ sin. } \phi$. (Art. 31.)

$$c \text{ sin. D}' \text{ sin. } \phi = a \text{ sin. D } \{\text{sin. } \alpha \text{ cos. } \phi - \text{cos. } \alpha \text{ sin. } \phi\}.$$

Dividing every term of this expression by sin. ϕ; and observing that $\frac{\text{cos. } \phi}{\text{sin. } \phi} = \text{cot. } \phi$ (Art. 25. *b.*),

$$c \text{ sin. D}' = a \text{ sin. D (sin. } \alpha \text{ cot. } \phi - \text{cos. } \alpha).$$

Whence, $a \text{ sin. D sin. } \alpha \text{ cot. } \phi = a \text{ sin. D cos. } \alpha + c \text{ sin. D}'$,

$$\text{and, cot. } \phi = \text{cot. } \alpha + \frac{c \text{ sin. D}'}{a \text{ sin. D sin. } \alpha}.$$

The angle ϕ being determined; $\text{BCD} = 360° - (\text{B} + \text{D} + \text{D}' + \phi)$ is known.

$$\text{Then DA} = \frac{\text{AB sin. (ADB + DAB)}}{\text{sin. ADB}} = \frac{c \text{ sin. (D} + \phi)}{\text{sin. D}};$$

$$\text{DB} = \frac{\text{AB sin. DAB}}{\text{sin. ADB}} = \frac{c \text{ sin. } \phi}{\text{sin. D}}.$$

$$\text{DC} = \frac{\text{BC sin. DBC}}{\text{sin. BDC}} = \frac{\text{BC sin. } \{180° - (\text{BCD} + \text{BDC})\}}{\text{sin. BDC}} = \frac{a \text{ sin. } (180° - 360° + \text{B} + \text{D} + \text{D}' + \phi - \text{D}')}{\text{sin. D}'} = \frac{a \text{ sin. } (\text{B} + \text{D} + \phi - 180°)}{\text{sin. D}'}$$

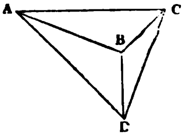
a. If $\text{B} = 180°$, or if the point B falls on the line AC, then $\alpha = 360° - 180° - (\text{D} + \text{D}')$;

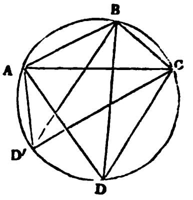
$$\text{or } a = 180° - (\text{D} + \text{D}'),$$

$$\therefore\ \text{cot. } \alpha = -\text{cot. (D} + \text{D}'), \text{ (Art. 25. } a.),$$
$$\text{sin. } \alpha = \text{sin. (D} + \text{D}'), \text{ (Art. 18. } b.),$$

$$\text{and cot. } \phi = -\text{cot. (D} + \text{D}') + \frac{c \text{ sin. D}'}{a \text{ sin. D sin. (D} + \text{D}')}.$$

b. If the point B falls below the line AC (as in the annexed figure), the sum of the angles ABD, CBD ($= 360° - \text{ABC}$) must be taken for the angle B: because these angles, together with the three remaining angles of the quadrilateral figure ABCD, make 360°, as has been supposed in the solution.





c. If ABC + ADC = 180°, or B + D + D′ = 180°

$$\alpha = 360° - 180° = 180°,$$

∴ cot. $\alpha = -\infty$. (Art. 25.) : sin. $\alpha = 0$. (Art. 18. *b.*)

∴ cot. $\phi = -\infty + \dfrac{c \text{ sin. D}'}{0}$; an expression from which the value of cot. ϕ cannot be determined.

Let a circle be described about the triangle ABC: then because the angles ABC, ADC, are together equal to two right angles, the point D is in the circumference of this circle (Euc. III. 22.): and wherever D, D′, &c. are taken in the circumference, ADB = AD′B; BDC = BD′C, &c. Consequently, under this hypothesis, the problem admits of an unlimited number of solutions.

d. In computing the value of cot. ϕ from the expression cot. $\phi = \text{cot. } \alpha + \dfrac{c \text{ sin. D}'}{a \text{ sin. D sin. } \alpha}$, it will be necessary to employ a table of natural tangents, or to find the natural from the logarithmic cotangent in the manner which is explained in Art. 55. *a.*

Examples.

1. Given $a = 3796$ ft.; $b = 3563.6$ ft.; $c = 2514$ ft.; D = 135° 19′; D′ = 113° 20′: required DA, DB, DC.

1st. ∠ B = 65° 7′ (Art. 59. Case IV.)

2d. $\alpha = 360° - (65° 7' + 135° 19' + 113° 20') = 360° - 313° 46' = 46° 14'$.

3d. Cot. $\phi = \text{cot. } 46° 14' + \dfrac{2514 \text{ sin. } 113° 20'}{3796 \text{ sin. } 135° 19' \text{ sin. } 46° 14'}$.

4th. Cot. 46° 14′ = .9578494.

5th.	Log. 2514	3.400365
	+ log. sin. 113° 20′	+ 9.962945
	+ arith. comp. log. 3796 . . .	+ 6.420674
	+ arith. comp. log. sin. 135° 19′ . .	+ 0.152929
	+ arith. comp. log. sin. 46° 14′ . .	+ 0.141365
	− 20	− 20.000000
The number of this log. is 1.197509 . . .		0.078278

Whence, in this instance, $\dfrac{c \text{ sin. D}'}{a \text{ sin. D sin. } \alpha} = 1.197509$.

6th. Cot. $\phi = .9578494 + 1.197509 = 2.1553584$.

7th. In the table of natural tangents, this number expresses the value of the cotangent of an angle of 24° 53′ 22″; ∴ $\phi = 24° 53' 22''$.

Otherwise,

to the log. of 2.1553584 = 0.333519
Add 10

The sum = 10.333519 is the log. cot. of 24° 53′ 22″.

8th. $\mathrm{DA}=\dfrac{c\ \sin.(\mathrm{D}+\phi)}{\sin.\mathrm{D}}=\dfrac{2514\times\sin.160^\circ\,12'\,22''}{\sin.135^\circ\,19'}=1210{\cdot}5$ feet.

9th. $\mathrm{DB}=\dfrac{c\ \sin.\phi}{\sin.\mathrm{D}}=\dfrac{2514\times\sin.24^\circ\,53'\,22''}{\sin.135^\circ\,19'}=1504.85$ feet.

10th. $\mathrm{DC}=\dfrac{a\ \sin.(\mathrm{B}+\mathrm{D}+\phi-180^\circ)}{\sin.\mathrm{D}'}=\dfrac{3796\times\sin.45^\circ\,19'22''}{\sin.113^\circ 20'}=2939.8$ feet.

2. Given $a=768$ ft.; $c=1490$ ft.; $\angle\mathrm{B}=125^\circ$, and B falling below AC (in which case $\angle\mathrm{B}=360^\circ-\mathrm{ABC}=235^\circ$); also $\mathrm{D}=37^\circ\ 10'$; $\mathrm{D}'=48^\circ\ 15'$ to find the angle ϕ.
Ans. $\phi=11^\circ\ 22'\ 39''$.

3. Given $a=849.43$ yds.; $c=1153.7$ yds.; $\mathrm{B}=112^\circ\ 25'$; $\mathrm{D}=27^\circ\ 31'$; $\mathrm{D}'=19^\circ\ 14'$: required ϕ.
Ans. $\phi=95^\circ\ 25'\ 42''$.

In this example, the result of the calculation is $\phi=-0.0950279$.

Whence, the sign being negative, ϕ is an obtuse angle. (Art. 25.) The angle to which the positive cot. 0.0950279 belongs is 84° 34′ 18″,

$$\therefore\ \phi=180^\circ-84^\circ\ 34'\ 18''=95^\circ\ 25'\ 42''.$$

e. The values of the unknown angles BAD, BCD, may be found, otherwise, as follows:

Let BCD be denoted by θ,

$$\text{Then, }\mathrm{BD}=\frac{\mathrm{AB}\ \sin.\mathrm{BAD}}{\sin.\mathrm{BDA}}=\frac{c\ \sin.\phi}{\sin.\mathrm{D}}:\ \text{also }\mathrm{BD}=\frac{\mathrm{BC}\ \sin.\mathrm{BCD}}{\sin.\mathrm{BDC}}=\frac{a\ \sin.\theta}{\sin.\mathrm{D}'},$$

$$\therefore\ \frac{a\ \sin.\theta}{\sin.\mathrm{D}'}=\frac{c\ \sin.\phi}{\sin.\mathrm{D}};\ \text{and }\sin.\theta=\frac{c\ \sin.\phi\ \sin.\mathrm{D}'}{a\ \sin.\mathrm{D}}.$$

To abridge this expression, let $\dfrac{\sin.\mathrm{D}'}{a\ \sin.\mathrm{D}}=\dfrac{1}{d}$,

$$\therefore\ \sin.\theta=\frac{c\ \sin.\phi}{d}\quad\ldots\ldots\ldots\ 1$$

Adding sin. ϕ to each side of equation 1,

$$\sin.\theta+\sin.\phi=\frac{c\ \sin.\phi}{d}+\sin.\phi=\frac{(c+d)\ \sin.\phi}{d};\quad\ldots\ 2$$

Subtracting sin. ϕ from each side of equation 1,

$$\sin.\,\theta-\sin.\,\phi=\frac{c\sin.\,\phi}{d}-\sin.\,\phi=\frac{(c-d)\sin.\,\phi}{d}\quad . \quad . \quad 3$$

Dividing equation 2 by equation 3;

$$\frac{\sin.\,\theta+\sin.\,\phi}{\sin.\,\theta-\sin.\,\phi}=\frac{(c+d)\sin.\,\phi}{d}\div\frac{(c-d)\sin.\,\phi}{d}=\frac{c+d}{c-d}\quad . \quad . \quad 4$$

But $\frac{\sin.\,\theta+\sin.\,\phi}{\sin.\,\theta-\sin.\,\phi}=\frac{\tan.\,\frac{1}{2}(\theta+\phi)}{\tan.\,\frac{1}{2}(\theta-\phi)}$; (Art. 40.)

$$\therefore\ \frac{\tan.\,\frac{1}{2}(\theta+\phi)}{\tan.\,\frac{1}{2}(\theta-\phi)}=\frac{c+d}{c-d}.$$

$$\therefore\ \tan.\,\tfrac{1}{2}(\theta-\phi)=\frac{(c-d)\tan.\,\frac{1}{2}(\theta+\phi)}{c+d}\quad . \quad . \quad . \quad . \quad . \quad 5$$

Now the sum of the angles θ and ϕ, ($=$ADC$-$ABC) is given: therefore, tan. $\frac{1}{2}(\theta+\phi)$ is given: consequently, tan. $\frac{1}{2}(\theta-\phi)$ is given by equation 5.

Therefore the angles θ and ϕ are given individually. (Art. 40.)

$$\text{Since } \frac{1}{d}=\frac{\sin.\,\text{D}'}{a\sin.\,\text{D}},\ d=\frac{a\sin.\,\text{D}}{\sin.\,\text{D}'}.$$

If the value of *d*, given by this expression, is greater than *c*, the value of sin. $\theta-$ sin. ϕ in equation 3 is negative: wherefore sin. ϕ is greater than sin. θ. In this case, equation 3 must be changed into

$$\sin.\,\phi-\sin.\,\theta=\frac{(d-c)\sin.\,\phi}{d},\ \text{and equation 5 into}$$

$$\tan.\,\tfrac{1}{2}(\phi-\theta)=\frac{(d-c)\tan.\,\frac{1}{2}(\phi+\theta)}{d+c}\quad . \quad . \quad . \quad . \quad . \quad 6$$

Employing Formula 5 or 6, to find the value of ϕ in Example 1., which has been already resolved by the other method,

$$d=\frac{a\sin.\,\text{D}}{\sin.\,\text{D}'}=\frac{3796\times\sin.\,135^\circ\,19'}{\sin.\,113^\circ\,20'}.$$

$$\therefore \text{Log. } d\left\{\begin{array}{ll}\text{log. } 3796\ .\ .\ .\ .\ .\ . & 3.579326\\ +\text{log. sin. } 135^\circ\,19'\ .\ .\ .\ .\ . & +9.847071\\ +\text{arith. comp. of log. sin. } 113^\circ\,20'\ . & +0.037055\\ -10\ .\ .\ .\ .\ .\ .\ . & -10.000000\end{array}\right.$$

The number of this log. is 2907.046 . . . . 3.463452

$$\therefore d=2907.046.$$

Since $c=2514$; $d>c$. Formula 6. is consequently to be used.

$$\text{Now, } d-c=2907.046-2514=393.046.$$
$$d+c=2907.046+2514=5421.046.$$

Also,

$$\text{ADC}=360^\circ-(\text{D}+\text{D}')=360^\circ-(135^\circ 19'+113^\circ 20')=360^\circ-248^\circ 39'=111^\circ 21',$$
$$\text{And ABC}=65^\circ\ 7',$$
$$\therefore \text{ADC}-\text{ABC}=111^\circ\ 21'-65^\circ\ 7'=46^\circ\ 14'=\phi+\theta,$$
$$\therefore \tfrac{1}{2}(\phi+\theta)=23^\circ\ 7'.$$
$$\text{And tan.}\tfrac{1}{2}(\phi-\theta)=\frac{393.046\times\text{tan. }23^\circ\ 7'}{5421.046}.$$

$$\therefore \text{log. tan.}\tfrac{1}{2}(\phi-\theta)=\left\{\begin{array}{lr}\text{log. } 393.046\ \ .\ \ .\ \ .\ \ . & 2.594444\\ +\text{log. tan. } 23^\circ\ 7'\ \ .\ \ .\ \ . & +9.630306\\ +\text{arith. comp. of log. } 5421.046\ \ . & +6.265917\\ -10\ .\ \ .\ \ .\ \ .\ \ .\ \ . & -10.000000\end{array}\right.$$

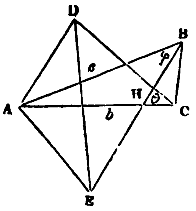
$$\text{The angle of this log. tan. being } 1^\circ\ 46'\ 22''\quad .\quad .\quad 8.490667$$

$$\tfrac{1}{2}(\phi-\theta)=1^\circ\ 46'\ 22'',$$
$$\therefore \phi=23^\circ\ 7'+1^\circ\ 46'\ 22''=24^\circ\ 53'\ 22''.\ \text{(Art. 40.)}$$
$$\theta=23^\circ\ 7'-1^\circ\ 46'\ 22''=21^\circ\ 20'\ 38''.$$

The angles ϕ and θ being found; the distances DA, DB, DC, are determined by Art. 59. Case I.

For a geometrical construction of this problem, see Practical Astronomy and Geodesy, Art. 425.

74. To determine the positions of two objects with respect to three stations whose mutual distances are known, when there have been observed, at the place of each object, the angles contained between a line supposed to join the objects, and other lines imagined to be drawn from them to two of the three given stations, some one of these being invisible from each object.



Let A, B, C, be the given stations, and D, E, the objects whose positions with respect to A, B, C, are to be determined.

Then there are given AB, BC, AC, and the angles ADE, EDC, AED, DEB, to find DA, DC, EA, EB, and DE.

Let $\text{BC}=a$, $\text{AC}=b$; $\text{AB}=c$; $\text{ADE}=\text{D}$; $\text{EDC}=\text{D}'$; $\text{AED}=\text{E}$; $\text{DEB}=\text{E}'$.

Since a, b, c, are given, the angles A, B, C, of the triangle ABC, are given, (Art. 59. Case IV.).

And since the angles DEB, EDC, are given, the angle DHE or BHC $(=180^\circ-(\text{D}'+\text{E}'))$ is given.

Again, since BHC and BAC, or A, are given, the sum of the angles ABE, ACD, which is equal to BHC$-$BAC, or to $180^\circ-(\text{A}+\text{D}'+\text{E}')$, is given.

Now, if the angles ABE, ACD were known,

In the triangle ADC, there would be given the side AC and the angles ADC, ACD to find DA, DC:

In the triangle ABE, there would be given the side AB and the angles AEB, ABE to find EA, EB:

And in the triangle ADE, there would be given the side AD or the side AE and the angles ADE, AED to find DE.

To determine the angles ABE, ACD; let ABE be denoted by ϕ, and ACD by θ:

Then, $\sin.\,\theta : \sin.\,(\mathrm{D}+\mathrm{D}') :: \mathrm{AD} : b$, (Art. 57.)

and, $\sin.\,\phi : \sin.\,(\mathrm{E}+\mathrm{E}') :: \mathrm{AE} : c$

$$\therefore \frac{\sin.\,\theta}{\sin.\,\phi} : \frac{\sin.\,(\mathrm{D}+\mathrm{D}')}{\sin.\,(\mathrm{E}+\mathrm{E}')} :: \frac{\mathrm{AD}}{\mathrm{AE}}\left(=\frac{\sin.\,\mathrm{E}}{\sin.\,\mathrm{D}}\right) : \frac{b}{c}$$

$$\therefore \frac{\sin.\,\theta}{\sin.\,\phi} = \frac{c \sin.\,\mathrm{E} \sin.\,(\mathrm{D}+\mathrm{D}')}{b \sin.\,\mathrm{D} \sin.\,(\mathrm{E}+\mathrm{E}')}.$$

To abridge this expression, let $\dfrac{\sin.\,\mathrm{E} \sin.\,(\mathrm{D}+\mathrm{D}')}{b \sin.\,\mathrm{D} \sin.\,(\mathrm{E}+\mathrm{E}')} = \dfrac{1}{d}$;

$$\text{then } \frac{\sin.\,\theta}{\sin.\,\phi} = \frac{c}{d}, \text{ and } \sin.\,\theta = \frac{c \sin.\,\phi}{d} \quad . \; . \; . \; . \; . \; 1$$

Adding $\sin.\,\phi$ to both sides of equation 1,

$$\sin.\,\theta + \sin.\,\phi = \frac{(c+d)\sin.\,\phi}{d} \quad . \; . \; . \; . \; . \; 2$$

Subtracting $\sin.\,\phi$ from both sides of equation 1,

$$\sin.\,\theta - \sin.\,\phi = \frac{(c-d)\sin.\,\phi}{d} \quad . \; . \; . \; . \; . \; 3$$

Dividing equation 2 by equation 3,

$$\frac{\sin.\,\theta + \sin.\,\phi}{\sin.\,\theta - \sin.\,\phi} = \frac{c+d}{c-d}. \quad . \; . \; . \; . \; . \; 4$$

Whence, as in Article 73. *e.*

$$\frac{\tan.\,\frac{1}{2}(\theta+\phi)}{\tan.\,\frac{1}{2}(\theta-\phi)} = \frac{c+d}{c-d}$$

$$\text{and } \tan.\,\tfrac{1}{2}(\theta-\phi) = \frac{(c-d)\tan.\,\frac{1}{2}(\theta+\phi)}{c+d} \quad . \; . \; . \; . \; . \; 5$$

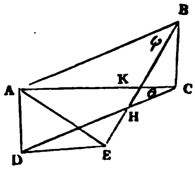
Consequently, as $\frac{1}{2}(\theta+\phi) = \frac{1}{2}\{180^\circ - (\mathrm{A}+\mathrm{D}'+\mathrm{E}')\}$, and $\frac{1}{2}(\theta-\phi)$ is given by equation 5, the values of θ and ϕ are determined.

Lastly, $\mathrm{DA}=\dfrac{b\sin.\theta}{\sin.(\mathrm{D}+\mathrm{D}')}$: $\mathrm{DC}=\dfrac{b\sin.(\mathrm{D}+\mathrm{D}'+\theta)}{\sin.(\mathrm{D}+\mathrm{D}')}$

$$\mathrm{EA}=\frac{c\sin.\phi}{\sin.(\mathrm{E}+\mathrm{E}')}:\ \mathrm{EB}=\frac{c\sin.(\mathrm{E}+\mathrm{E}'+\phi)}{\sin.(\mathrm{E}+\mathrm{E}')}$$

$$\mathrm{DE}=\frac{\mathrm{DA}\sin.(\mathrm{D}+\mathrm{E})}{\sin.\mathrm{E}}=\frac{b\sin.\theta\sin.(\mathrm{D}+\mathrm{E})}{\sin.(\mathrm{D}+\mathrm{D}')\sin.\mathrm{E}}.$$

This method of solution is, in general, applicable, whether the objects D, E, are or are not situated with respect to the stations A, B, C, as in the diagram. But when the positions of D, E, are different, the signs of some of the quantities are different. For example, if the line joining the objects D, E, falls wholly without the triangle ABC, as in the annexed figure,



Then if $\mathrm{ADE}=\mathrm{D}$ and $\mathrm{EDC}=\mathrm{D}'$, $\mathrm{ADC}=\mathrm{D}-\mathrm{D}'$, not $\mathrm{D}+\mathrm{D}'$ as before: and if $\mathrm{AED}=\mathrm{E}$, and $\mathrm{DEB}=\mathrm{E}'$, $\mathrm{AEB}=\mathrm{E}'-\mathrm{E}$, not $\mathrm{E}+\mathrm{E}'$, as before.

Also from the angle $\mathrm{BHC}=180°-(\mathrm{D}'+\mathrm{E}')$ and $\mathrm{BAC}=\mathrm{A}$, $\theta-\phi$ is found, and not $\theta+\phi$ as before: for $\mathrm{BHC}+\mathrm{HKC}+\theta=180°$ (Euc. I. 32.), and $\mathrm{BAC}+\mathrm{HKC}+\phi=180°$,

$$\therefore\ \mathrm{BHC}-\mathrm{BAC}=\theta-\phi;$$

or, since $\mathrm{BHC}=180°-(\mathrm{D}'+\mathrm{E}')$ and $\mathrm{BAC}=\mathrm{A}$,

$$180°-(\mathrm{A}+\mathrm{D}'+\mathrm{E}')=\theta-\phi.$$

For a geometrical construction of this problem see Practical Astronomy and Geodesy, Art. 426.

Ex. Given $a=2678$ yards; $b=4490$ yards; $c=5232$ yards; $\mathrm{D}=52°\,58'$; $\mathrm{D}'=55°\,4'$; $\mathrm{E}=41°\,46'$; $\mathrm{E}'=38°\,4'$: required DA, DC, EA, EB, DE.

1st. $\angle\mathrm{A}=30°\,47'$. (Art. 59. Case IV.)

2d. $d=\dfrac{b\sin.\mathrm{D}\sin.(\mathrm{E}+\mathrm{E}')}{\sin.\mathrm{E}\sin.(\mathrm{D}+\mathrm{D}')}=\dfrac{4490\sin.52°\,58'\sin.79°\,50'}{\sin.41°\,46'\sin.108°\,2'}=5570.16.$

$$\therefore d+c=5570.16+5232=10802.16.$$
$$d-c=5570.16-5232=\quad 338.16.$$

3d. $\phi+\theta=180°-(\mathrm{A}+\mathrm{D}'+\mathrm{E}')=180°-(30°\,47'+55°\,4'+38°\,4')=180°-123°\,55'=56°\,5'$,

$$\therefore\tfrac{1}{2}(\phi+\theta)=28°\,2'\,30''.$$

4th. $\mathrm{Tan.}\tfrac{1}{2}(\phi-\theta)=\dfrac{(d-c)\tan.\frac{1}{2}(\phi+\theta)}{d+c}=\dfrac{338.16\times\tan.28°\,2'\,30''}{10802.16}=\mathrm{tan.\,of}\ 57'\,20''$,

$$\therefore\phi=28°\,2'\,30''+57'\,20''=28°\,59'\,50'',$$
$$\theta=28°\,2'\,30''-57'\,20''=27°\,5'\,10''.$$

5th. $\text{DA} = \dfrac{b \sin.\theta}{\sin.(\text{D} + \text{D}')} = \dfrac{4490 \times \sin.27^\circ 5' 10''}{\sin.108^\circ 2'} = 2150.04$ yards.

6th. $\text{DC} = \dfrac{b \sin.(\text{D} + \text{D}' + \theta)}{\sin.(\text{D} + \text{D}')} = \dfrac{4490 \times \sin.135^\circ 7' 10''}{\sin.108^\circ 2'} = 3331.96$ yds.

7th $\text{EA} = \dfrac{c \sin.\phi}{\sin.(\text{E} + \text{E}')} = \dfrac{5232 \times \sin.28^\circ 59' 50''}{\sin.79^\circ 50'} = 2576.76$ yards.

8th. $\text{EB} = \dfrac{c \sin.(\text{E} + \text{E}' + \phi)}{\sin.(\text{E} + \text{E}')} = \dfrac{5232 \times \sin.108^\circ 49' 50''}{\sin.79^\circ 50'} = 5030.95$ yards.

9th. $\text{DE} = \dfrac{\text{DA} \sin.(\text{D}+\text{E})}{\sin.\text{E}} = \dfrac{b \sin.\theta \sin.(\text{D}+\text{E})}{\sin.(\text{D}+\text{D}') \sin.\text{E}} = \dfrac{4490 \times \sin.27^\circ 5' 10'' \times \sin.94^\circ 44'}{\sin.108^\circ 2' \times \sin.41^\circ 46'} = 3216.8$ yards.

SECTION VIII.

MENSURATION OF PLANE SURFACES BOUNDED BY STRAIGHT LINES AND BY THE CIRCUMFERENCES OF CIRCLES.

75. To find the area of a parallelogram.

Given the base and altitude of a parallelogram to find its area.

Let b, a, the numerical values of the base and altitude of a parallelogram be given; and let its area be denoted by s: then by Art. 7. *e.*,

$$\text{s} = ba:$$

in words: — The area of a parallelogram is expressed by the product of the numerical values of the base and altitude of the parallelogram.

If the parallelogram is rectangular, a expresses the numerical value of either of the sides adjacent to the base: whence the area of a rectangle is obtained by forming the product of any two adjacent sides of the rectangle.

If the parallelogram is rectangular and equilateral, $a = b$; the figure is a square, and its area is obtained by forming the square of the numerical value of one of its sides.

Examples. 1st. Let $b = 15.5$; $a = 13.42$: required s.
Ans. $\text{s} = 208.01$.

2d. Let $b = 14$ ft. 6 in.; $a = 4$ ft. 9 in.: required s.
Ans. $\text{s} = 68$ sq. ft. 126 sq. in.*

3d. Let $b = 2$ ft. 10 in. 6 li.; $a = 9$ in.: required s.
Ans. $\text{s} = 2$ sq. ft. $22\frac{1}{2}$ sq. in.

4th. If the lengths of two adjacent sides of a rectangular field are 1375 links and 950 links respectively, what is the area of the field in acres, &c.?
Ans. 13 ac. 0 ro. 10 poles.

5th. The length of a rectangle is $19\frac{1}{2}$ feet, and the breadth $10\frac{1}{5}$ feet; what is the area?
Ans. 198.9 square feet.

---

* For the multiplication of duodecimal numbers, see Appendix, Art. 14.

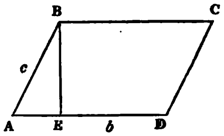
6th. What is the area of a square whose side is 37.25 chains in length?

Ans. 138 ac. 3 ro. 1 pole.

a. Given two sides of a parallelogram and the angle included between them, to find the area of the parallelogram.

Let AC be any parallelogram, and let the sides $\text{AD} = b$, $\text{AB} = c$, and the angle $\text{BAD} = \text{A}$ be given to find S, the area of the parallelogram.

From B let BE be drawn perpendicular to AD:



$$\text{then } \text{S} = b \times \text{BE}, \text{ (Art. 7. } e.)$$
$$\text{also, } \text{BE} = c \text{ sin. A}, \text{ (Art. 56.)}$$
$$\therefore \text{ S} = bc \text{ sin. A}.$$

Whence the area of a parallelogram, expressed in terms of two adjacent sides and the angle included between them, is obtained by multiplying the product of the two given sides by the sine of the included angle.

The calculation of S from the expression bc sin. A, may be made with the help of a table of natural sines, or by logarithms.

If by the latter method, since $\text{S} = bc \text{ sin. A}$,

$$\text{log. S} = \text{log. } b + \text{log. } c + \text{log. sin. A} - 10.$$

Examples. 1st. Given $b = 35.63$; $c = 25.79$; $\text{A} = 25^\circ\ 13'$: required S.

$$\text{Log. S} = \begin{cases} \text{log. } 35.63 \ldots\ldots & 1.551816 \\ +\text{log. } 25.79 \ldots\ldots & +1.411451 \\ +\text{log. sin. } 25^\circ\ 13' \ldots & +9.629453 \\ -10 \ldots\ldots & -10.000000 \end{cases}$$

The number of this log. is 391.49 . . 2.592720

$$\therefore \text{S} = 391.49.$$

2d. Given $b = 40$; $c = 30$; $\text{A} = 28^\circ\ 57'\ 18''$: required S.

Ans. $\text{S} = 580.947$.

3d. Given $b = 138$ ft.; $c = 59$ ft.; $\text{A} = 17^\circ\ 56'\ 40''$: required S.

Ans. $\text{S} = 2508.5$ sq. ft.

76. To find the area of a triangle.

Given the base and altitude of a triangle to find its area.

If the values of the given base and altitude are b, a, and the area (which is to be found) is denoted by S,

$$\text{S} = \frac{ba}{2}, \text{ (Art. 7. } f.):$$

in words: — The area of a triangle is obtained by forming the product of the numerical values of the base and altitude of the triangle, and taking the half of this product.

If the proposed triangle is right-angled, the perpendicular is equal to a; the area of a right-angled triangle is consequently obtained by multiplying the base by the perpendicular and taking half the product.*

Examples. 1st. Given $b = 28.2$ yds.; $a = 18.4$ yds.: required S.

Ans. $S = 259.44$ sq. yds.

2d. Given $b = 18$ ft. 4 in.; $a = 11$ ft. 10 in.: required S.
Ans. $S = 108$ sq. ft. 68 sq. in.

3d. Given the base of a right-angled triangle $= 731$ links, and the perpendicular $= 618$ links: required the area of the triangle?

Ans. 2 ac. 1 ro. 1.4064 po.

a. Given two sides of a triangle and the angle included between them, to find the area of the triangle.

Let the two sides b, c, and the included angle A, be given to find the area of the triangle.

The area of the parallelogram of which two adjacent sides and the included angle are b, c, A, is given by the expression, bc sin. A, (Art. 75. *a.*).

And the parallelogram is double the triangle, (Euc. I. 41.)

$$\therefore \; S = \frac{bc \sin. A}{2}.$$

The rule for finding the area of any triangle of which two sides and the included angle are given, is therefore, — Multiply the product of the two sides by the sine of the included angle, and take half the result for the area of the triangle.

If the calculation is made with logarithms, the formula to be used is,

$$\text{Log. } S = \text{log. } b + \text{log. } c + \text{log. sin. A} + \text{arith. comp. of log. } 2 - 20.$$

Examples. 1st. Given $b = 25$ ft.; $c = 21.25$ ft.; $A = 45°$: required S.
Ans. $S = 187.8250312$ sq. ft.

Calculation of this example by natural sines: —

$$\text{Log. sin. } 45° = 9.849485 - 10. = \bar{1}.849485.$$

The number of the log. $\bar{1}.849485$ being .707106,
the natural sine of 45° is .707106.

---

* Here, and in all subsequent instances, by the product of two or more lines is meant the product of the numerical values of these lines.

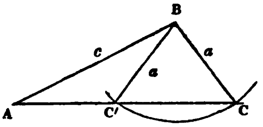
$$\frac{25 \times 21.25 \times .707106}{2} = \frac{531.25 \times .707106}{2} = \frac{375.6500625}{2} = 187.8250312 \text{ sq. feet.}$$

which is the area of the proposed triangle.

2d. Given $b = 95.543$ yds.; $c = 6.73$ yds.; $A = 121° 5'$: required S.

Ans. $S = 275.34$ sq. yds.

b. Given two sides and the angle opposite one of these sides, to find the area of the triangle.



Given c, a, A, to find the area ABC or S.

It appears from Article 59. Case II., that if $a > c$, there is only one triangle of which these data are three parts: but that if $a < c$, there are two, the one having an acute, and the other an obtuse angle subtended by the given side c.

Now the value or values of C are determined by Article 59. Case II.: and the value or values of B, from the formula $B = 180° - (A + C)$.

This problem is therefore reduced to that in which two sides and the included angle are given to find the area of the triangle.

$$\therefore \text{ Area ABC} = \frac{ac \sin. ABC}{2}.$$

$$\text{Area ABC}' = \frac{ac \sin. ABC'}{2}.$$

c. Given two angles of a triangle and one side, to find the area of the triangle.

Let the given angles and side be A, C, b; and let the area be denoted by S.

Then, since A, C, are given; $B = 180° - (A + C)$ is given.

$$\therefore \ c = \frac{b \sin. C}{\sin. B}, \text{ can be determined, (Art. 59. Case I.).}$$

$$\text{Now, } S = \frac{bc \sin. A}{2} \text{ or } c \times \frac{b \sin. A}{2}. \quad \text{(Art. 76. } a.\text{)}$$

Whence, by substituting the value of c,

$$S = \frac{b \sin. C}{\sin. B} \times \frac{b \sin. A}{2} = \frac{b^2 \sin. A \sin. C}{2 \sin. B}$$

$$\text{Or, since } \sin. B = \sin. (A + C), \ S = \frac{b^2 \sin. A \sin. C}{2 \sin. (A + C)}.$$

Employing logarithms, the formula to be used is,

$$\text{Log. S} = 2\,\text{log.}\,b + \text{log. sin. A} + \text{log. sin. C} + \text{arith. comp. of log. } 2 + \text{arith. comp. of log. sin. B} - 30.$$

Ex. Given $\text{A} = 38° \, 40'$; $\text{C} = 57° \, 50'$; $a = 120$ yds.: required S.

Ans. $\text{S} = 9692.3$ sq. yds.

d. Given the three sides of a triangle to find the area of the triangle.

The formula for the solution of this problem has been investigated in a preceding article, (Art. 8. *a.*). a, b, c, expressing the numerical values of the sides, s denoting $\frac{a+b+c}{2}$ and S the required area, the formula is,

$$\text{S} = \sqrt{s(s-a)(s-b)(s-c)}.$$

This formula translated into words is as follows: — The area of a triangle is expressed by the square root of the continued product of four numbers; which numbers are the semiperimeter of the triangle, and the excess of the semiperimeter above each of the three sides of the triangle.

If a, b, c, are expressed by large numbers, it may be convenient to employ logarithms in computing the value of S. The formula for this purpose is,

$$\text{log. S} = \tfrac{1}{2}\,\{\text{log.}\,s + \text{log.}\,(s-a) + \text{log.}\,(s-b) + \text{log.}\,(s-c)\}.$$

Examples. 1st. Given $a = 20.5$; $b = 43.3$; $c = 31.2$: required S.
Ans. $\text{S} = 296.31056$.

Calculation, with common numbers: —

$$s = \frac{20.5 + 43.3 + 31.2}{2} = \frac{95}{2} = 47.5.$$

$$s - a = 47.5 - 20.5 = 27.$$
$$s - b = 47.5 - 43.3 = 4.2.$$
$$s - c = 47.5 - 31.2 = 16.3.$$

$$\therefore \text{S} = \sqrt{47.5 \times 27 \times 4.2 \times 16.3} = \sqrt{87799.95} = 296.31056.$$

Calculation with logarithms: —

$$\text{Log. S} = \tfrac{1}{2}\left\{\begin{array}{lr} \text{log. } 47.5 \;\; . \;\; . & 1.676694 \\ + \text{log. } 27 \;\; . \;\; . \;\; . & +\,1.431364 \\ + \text{log. } 4.2 \;\; . \;\; . \;\; . & +\,0.623249 \\ + \text{log. } 16.3 \;\; . \;\; . & +\,1.212188 \\ \hline & 2)4.943495 \\ \hline \end{array}\right.$$

The number of this log. is 296.3106 . 2.4717475

$$\therefore \text{S} = 296.3106.$$

2d. Given $a = 41$ ft.; $b = 29$ ft.; $c = 56$ ft.: required S.
Ans. $\text{S} = 574.34$ sq. ft.

3d. Given $a = 563$ yds.; $b = 295$ yds.; $c = 387$ yds.: required s.

Ans. $s = 53447.73$ sq. yds.

4th. Given $a = 49$ chains; $b = 50.25$ chains; $c = 25.69$ chains: required s.

Ans. $s = 61$ ac. 1 ro. 39.68 poles.

e. It is found in the investigation, Article 18. *a.*, that

$4b^2x^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c)$, and that $\frac{bx}{2} = s$ (the area of the triangle)

$$\therefore 2s = bx, \text{ and } 16s^2 = 4b^2x^2.$$

$$\therefore 16s^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c),$$

$$\text{And } s = \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}.$$

Making $a = b$; $s = \frac{1}{4}\sqrt{(2a+c)(c)(c)(2a-c)} = \frac{1}{4}\sqrt{c^2(2a+c)(2a-c)} = \frac{c}{4}\sqrt{(2a+c)(2a-c)}$.

Making $a = b = c$; $s = \frac{1}{4}\sqrt{3a \times a \times a \times a} = \frac{1}{4}\sqrt{3a^4} = \frac{a^2}{4}\sqrt{3}$.

These formulæ may be employed to find the areas of isosceles and equilateral triangles whose sides are given.

77. To find the area of a trapezoid.

Given the parallel sides of a trapezoid and the perpendicular distance between them, to find the area of the trapezoid.

Let b, b' be the parallel sides, a, the altitude or perpendicular distance between the sides, and s, the required area; then,

$$s = \frac{b+b'}{2} \times a, \text{ (Euc. I. B.)}$$

Examples. 1st. Given $b = 12$ ft. 6 in.; $b' = 18$ ft. 4 in.; $a = 7$ ft. 9 in.: required s. *Ans.* $s = 119$ sq. ft. 69 sq. in.

CALCULATION.

$$\frac{b+b'}{2} = \frac{12\text{ ft. }6\text{ in.} + 18\text{ ft. }4\text{ in.}}{2} = \frac{30\text{ ft. }10\text{ in.}}{2} = 15\text{ ft. }5\text{ in.}$$

$$\therefore \frac{b+b}{2} \times a = (15\text{ ft. }5\text{ in.}) \times (7\text{ ft. }9\text{ in.}) = 119\text{ sq. ft. }69\text{ sq. in.} = s.$$

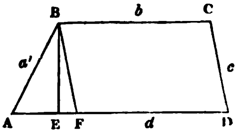
2d. Given $b = 10\frac{1}{2}$ in.; $b' = 16\frac{1}{4}$ in.; $a = 8\frac{1}{3}$ in.: required s.

Ans. $s = 109.675$ sq. inches.

3d. The lengths of the parallel sides of a trapezoidal field are 25.75 chains and 18.87 chains, respectively; and the perpendicular distance of the parallel sides is 10.54 chains: required the area of the field.

Ans. 23 ac. 2 ro. 2.3584 poles.

a. Given all the sides of a trapezoid to find its area.



In the trapezoid ABCD, let BC be parallel to AD: and let the values of the sides be given, viz. $AB=a'$, $BC=b$, $CD=c$, $AD=d$.

From B let BE be drawn perpendicular to AD, (Euc. I. 11.) and BF parallel to CD, (Euc. I. 31.)

Then $BF=CD=c$, and $FD=BC=b$ (Euc. I. 34.) $\therefore AF=AD-FD=d-b$: let $d-b=e$.

The sides a', c, e, of the triangle ABF being known,

$$\sin. A=\frac{2}{a'e}\sqrt{s(s-a')(s-c)(s-e)}, \text{ (Art. 57. } g.)$$

$$\therefore BE \text{ or } a' \sin. A=\frac{2}{e}\sqrt{s(s-a')(s-c)(s-e)}$$

$$\therefore ABCD \text{ or } \frac{b+d}{2}\times BE=\frac{b+d}{e}\sqrt{s(s-a')(s-c)(s-e)}.$$

Ex. Given $a'=122$ ft., $b=324$ ft., $c=137$ ft., $d=431$ ft.: required S.

Ans. $S=44079.77$ sq. ft.

b. Given the altitude of a trapezoid, one of the parallel sides, and the two angles adjacent to this side, to find the area of the trapezoid.

In the trapezoid AC (see the figure of the last problem), let BC be parallel to AD, and let BE be the perpendicular distance between BC, AD: then there are given $AD=d$, $BE=a$, and the angles A, D, to find S.

BF being drawn parallel to CD, $BC=FD$, (Euc. I. 34.), and $\angle BFA=D$, (Euc. I. 29.).

$$\text{Now, } BA=\frac{BE}{\sin. A} \text{ (Art. 56.)}=\frac{a}{\sin. A},$$

$$\text{And } AF=BA\cdot\frac{\sin.(A+D)}{\sin. D} \text{ (Art. 57.)}=\frac{a}{\sin. A}\times\frac{\sin.(A+D)}{\sin. D}=\frac{a\sin.(A+D)}{\sin. A\sin. D},$$

$$\therefore FD \text{ or } AD-AF=d-\frac{a\sin.(A+D)}{\sin. A\sin. D}=BC,$$

$$\therefore AD+BC=2d-\frac{a\sin.(A+D)}{\sin. A\sin. D}, \text{and } \frac{AD+BC}{2}=d-\frac{a\sin.(A+D)}{2\sin. A\sin. D},$$

$$\therefore \frac{AD+BC}{2}\times a=S=da-\frac{a^2\sin.(A+D)}{2\sin. A\sin. D}.$$

If BE, the altitude of the trapezoid, BC, the shorter of the two parallel sides, and B, C, the angles adjacent to BC, are given to find S,

$\angle\,\mathrm{A}+\mathrm{ABC}$, or $\mathrm{B}=180°$, (Euc. I. 29.) $\therefore \sin.\mathrm{A}=\sin.\mathrm{B}$, (Art. 18.)

$$\mathrm{AFB}=\mathrm{ADC}:\ \text{and}\ \mathrm{ADC}+\mathrm{C}=180° \quad \therefore\ \sin.\mathrm{AFB}=\sin.\mathrm{C}.$$

$$\mathrm{FBC}+\mathrm{C}=180° \quad \therefore\ \mathrm{ABF}+\mathrm{FBC}+\mathrm{C}=\mathrm{ABF}+180°,\ \text{or}\ \mathrm{ABC}+\mathrm{C}=\mathrm{ABF}+180°.$$

$$\therefore\ \sin.\mathrm{ABF}=\sin.(\mathrm{ABC}+\mathrm{C})=\sin.(\mathrm{B}+\mathrm{C}),\ (\text{Art. 19.})$$

$$\text{Now}\ \mathrm{BA}=\frac{\mathrm{BE}}{\sin.\mathrm{A}},\ (\text{Art. 56.})=\frac{\mathrm{BE}}{\sin.\mathrm{B}}=\frac{a}{\sin.\mathrm{B}},$$

$$\text{and}\ \mathrm{AF}=\mathrm{BA}\frac{\sin.\mathrm{ABF}}{\sin.\mathrm{AFB}}=\frac{a}{\sin.\mathrm{B}}\times\frac{\sin.(\mathrm{B}+\mathrm{C})}{\sin.\mathrm{C}}=\frac{a\sin.(\mathrm{B}+\mathrm{C})}{\sin.\mathrm{B}\sin.\mathrm{C}},$$

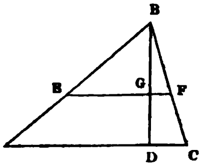
$$\therefore\ \mathrm{BC}+\mathrm{AD}=2\mathrm{BC}+\mathrm{AF}=2b+\frac{a\sin.(\mathrm{B}+\mathrm{C})}{\sin.\mathrm{B}\sin.\mathrm{C}},$$

$$\text{And,}\ \frac{\mathrm{BC}+\mathrm{AD}}{2}\times a=ba+\frac{a^2\sin.(\mathrm{B}+\mathrm{C})}{2\sin.\mathrm{B}\sin.\mathrm{C}}.$$

Ex. Given $d=350$; $a=34$; $\mathrm{A}=37°\,18'$; $\mathrm{D}=52°\,42'$: required S.

Ans. $\mathrm{S}=10700.95$.

78. From a given triangle to cut off a trapezoid which shall contain a given area.



1st. Let the base, b, and altitude a, of the triangle ABC be given; and let it be required to find values of EF or y, which is parallel to AC, and of DG or x, which is perpendicular to AC, such that the trapezoid AEFC shall contain a given area $=\mathrm{S}$.

Since EF is parallel to AC, $\mathrm{AC}:\mathrm{EF}::\mathrm{DB}:\mathrm{GB}$,

$\therefore\ \mathrm{AC}:\mathrm{AC}-\mathrm{EF}::\mathrm{DB}:\mathrm{DB}-\mathrm{GB}$, (Propor. Prop. 7.)
or, $\mathrm{AC}:\mathrm{AC}-\mathrm{EF}::\mathrm{DB}:\mathrm{DG}$

$$\text{or,}\ b\ :\ b-y::a\ :\ x\quad\therefore\ x=\frac{a}{b}(b-y)\quad .\ .\ 1$$

$$\text{And since, by the problem,}\ \frac{b+y}{2}\times x=\mathrm{S}\ .\ .\ .\ x=\frac{2\mathrm{S}}{b+y}\quad .\ .\ 2$$

$$\therefore\ (\text{Eq. 1, 2.})\ \frac{a}{b}(b-y)=\frac{2\mathrm{S}}{b+y},\ \text{or}\ b^2-y^2=\frac{2b\mathrm{S}}{a}.$$

$$\therefore\ y=\sqrt{\left(b^2-\frac{2b\mathrm{S}}{a}\right)}$$

$$x=\frac{a}{b}\left\{b-\sqrt{\left(b^2-\frac{2b\mathrm{S}}{a}\right)}\right\}.$$

Examples. 1st. Given the base of a triangle $= 20$ ft.; the altitude $= 18$ ft.; the trapezoidal area to be cut off $= 80$ sq. ft.: required the length of the line of section and its distance from the base of the triangle.
Ans. $y = 14.9071$ ft.; $x = 4.58361$ ft.

CALCULATION.

$$b = 20;\ a = 18;\ s = 80.$$

$$\therefore b^2 = 400;\ \frac{2bs}{a} = \frac{2 \times 20 \times 80}{18} = \frac{1600}{9}:\ b^2 - \frac{2bs}{a} = 400 - \frac{1600}{9} = \frac{2000}{9},$$

$$\therefore y = \sqrt{\left(\frac{2000}{9}\right)} = \frac{44.7213}{3} = 14.9071;$$

$$x = \frac{18}{20}(20 - 14.9071) = \frac{9}{10} \times 5.0929 = 4.58361.$$

2d. Let the base, b, the angles, A, C, and the area, S, be given, to find y, x.

Then, $\text{AB} = \dfrac{b \sin. \text{C}}{\sin. (\text{A} + \text{C})}$ (Art. 57.): and $\text{BD} = \text{AB} \sin \text{A}$. (Art. 56.)

$$\therefore \text{BD} = \frac{b \sin. \text{A} \sin. \text{C}}{\sin. (\text{A} + \text{C})} = \text{altitude of the triangle ABC} = a.$$

Substituting this expression for a in the values of y, x, (1st Case),

$$y = \sqrt{\left\{b^2 - 2\,\text{S} \times \frac{\sin. (\text{A} + \text{C})}{\sin. \text{A} \sin. \text{C}}\right\}},$$

$$x = \frac{\sin. \text{A} \sin. \text{C}}{\sin. (\text{A} + \text{C})}\left\{b - \sqrt{\left(b^2 - 2\,\text{S} \times \frac{\sin. (\text{A} + \text{C})}{\sin. \text{A} \sin. \text{C}}\right)}\right\}.$$

If $\text{A} = \text{C}$, $\dfrac{\sin. \text{A} \sin. \text{C}}{\sin. (\text{A} + \text{C})} = \dfrac{\sin.^2 \text{A}}{\sin. 2\,\text{A}} = \dfrac{\sin.^2 \text{A}}{2 \sin. \text{A} \cos. \text{A}} = \dfrac{\sin. \text{A}}{2 \cos. \text{A}} = \dfrac{1}{2} \tan. \text{A}$ (Art. 19. *c.*),

$$\frac{\sin. (\text{A} + \text{C})}{\sin. \text{A} \sin. \text{C}} = \frac{2 \cos. \text{A}}{\sin. \text{A}} = 2 \cot. \text{A},\ (\text{Art. } 25.\ b.)$$

Whence, in this case, $\text{C} = \text{A}$, $y = \sqrt{(b^2 - 4\,\text{S} \times \cot. \text{A})}$,
$$x = \tfrac{1}{2} \tan. \text{A}\ \{b - \sqrt{(b^2 - 4\,\text{S} \times \cot. \text{A})}\}.$$

Ex. Given $b = 20$ ft.; $\text{A} = 56° 19'$; $\text{C} = 63° 26'$; $\text{S} = 109$ sq. ft.: required y and x.
Ans. $y = 12.07047$ ft.; $x = 6.79753$ ft.

CALCULATION.

$$\text{A} + \text{C} = 119° 45',$$

$\therefore \log. \left(\dfrac{\sin. \text{A} \sin. \text{C}}{\sin. (\text{A} + \text{C})}\right) = \log. \sin. 56° 19' - 10 + \log. \sin. 63° 26' - 10 + \text{arith. comp. of} \log. \sin. 119° 45'$.

log. sin. 56° 19' 9.920184
+ log. sin. 63° 26' . . . . + 9.951539
+ arith. comp. of log. sin. 119° 45' + 0.061381
− 20 . . . . − 20.000000

The number of this log. is .85724 . . $\bar{1}\cdot933104$

Consequently the numerical value of $\frac{\sin. A \sin. C}{\sin. (A+C)}$ is .85724,

and that of $\frac{\sin. (A+C)}{\sin. A \sin. C} = \frac{1}{.85724}$:

also, $b^2 = 400$; $2S = 218$;

$$\therefore y = \sqrt{\left(400 - \frac{218}{.85724}\right)} = \sqrt{(400 - 254.3036)} = \sqrt{(145.4964)} = 12.07047,$$

$$x = .85724(20 - 12.07047) = .85724 \times 7.92953 = 6.79753.$$

VERIFICATION.

$$S = \frac{20 + 12.07047}{2} \times 6.79753 = 109.000296 + .$$

Examples in both cases.

Ex. 1. Given $b = 36$ ft.; $a = 28$ ft.; $S = 252$ sq. ft.: required y and x.

Ans. $y = 25.457$ ft.; $x = 8.201$ ft.

Ex. 2. Given $b = 3$ ft. 4 in.; $a = 2$ ft. 8 in.; $S = 2\frac{1}{2}$ sq. ft.: required y and x.

Ans. $y = 2$ ft. 2.4575 in.; $x = 10.834$ in.

Ex. 3. In a perpendicular section of a ditch, given the breadth at top = 30 ft., the slopes of the sides (or inclinations of the sides to the plane of the horizon) = 65° 25′ and 70° 45′; and the area of the section = 260 sq. ft.: required the breadth at bottom and the depth.

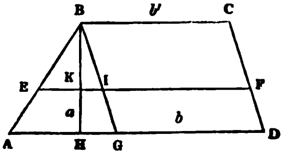
Ans. Breadth at bottom = 21.9206 feet; depth = 10.016 feet.

Ex. 4. In a perpendicular section of a ditch, given the breadth at top = 26 ft., the slopes of the sides each = 45°, and the area = 140 sq. ft.: required the breadth at bottom, and the depth of the ditch.

Ans. Breadth = 10.77 ft.; depth = 7.615 ft.

79. From a given trapezoid to cut off, by a line parallel to the parallel sides, a part which shall contain a given area.

In the trapezoid AC, let the parallel sides $AD = b$, $BC = b'$, and the altitude $BH = a$, be given; and let it be assumed that EF, which is parallel to AD, BC, cuts off from AC a given area S.



Draw BG parallel to CD (Euc. i. 31.). Let EF cut BH in K: also let BK be denoted by x, and EF by y.

Then, since EI is parallel to AG,

AG : EI (:: BG : BI) :: BH : BK, (Euc. VI. 4.).

But AG $=b-b'$; EI $=y-b'$; BH $=a$; BK $=x$,

$$\therefore\ b-b' : y-b' :: a : x,$$

$$\therefore\ (b-b')x=a(y-b') \quad . \quad . \quad . \quad 1$$

Next, area EC or $s=\frac{\text{EF}+\text{BC}}{2}\times\text{BK}=\frac{1}{2}(y+b')x$,

$$\therefore\ (y+b')x=2s \quad . \quad . \quad . \quad . \quad 2$$

From equations 1, 2 (Alg. Art. 130.), the values of x, y, are found to be

$$y=\sqrt{\left\{\frac{2s(b-b')}{a}+b'^2\right\}}$$

$$x=\frac{a}{b-b'}\left\{-b'+\sqrt{\left(\frac{2s(b-b')}{a}+b'^2\right)}\right\}.$$

The value of y gives the length of the line of section (EF); and the value of x gives the distance (BK) of the line of section from the point B.

If $s>$ AC, $x>a$: in this case EF intersects the lines BA, BH, BG, and CD, below the base AD.

Examples. 1st. Given $b=20$ in.; $b'=14$ in.; $a=12$ in., and therefore AC $=204$ sq. in.: it is required to cut from the trapezoid AC a part containing 144 sq. in: Required the length of the line of section and its distance from the side BC, to which it is parallel.

Ans. $y=18.439$ in.; $x=8.878$ in.

CALCULATION.

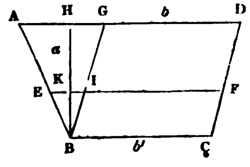
$$y=\sqrt{\left\{\frac{2\times144(20-14)}{12}+14^2\right\}}=\sqrt{\left\{\frac{2\times144\times6}{12}+14\times14\right\}}=\sqrt{\{144+196\}}=\sqrt{340}=18.439,$$

$$x=\frac{12}{20-14}(-14+18.439)=2\times4.439=8.876.$$

2d. Given $b=76$ ft., $b'=36$ ft., $a=23$ ft., $s=560$ sq. ft.: required y and x.

Ans. $y=56.954$ ft.; $x=12.04855$ ft.

a. In the last problem, the part EC, whose area is s, is cut from the smaller end of the given trapezoid AC. If the part which is to contain an area $=s$, is required to be cut from the greater end of AC, the formulæ for determining the values of the line of section, EF or y, and the altitude HK or x, require modification.



Let it be supposed that EF, which is parallel to AD or BC, cuts from the greater end of AC a trapezoid ED whose area is S.

Also let AD $=b$, BC $=b'$, HB $=a$, HK $=x$, EF $=y$; and therefore AG $=b-b'$, EI $=y-b'$. Then, as in the last problem,

$$\text{AG} : \text{EI} :: \text{HB} : \text{KB},$$
$$\therefore \text{AG} : \text{AG}-\text{EI} :: \text{HB} : \text{HB}-\text{KB},$$
$$\text{or AG} : \text{AG}-\text{EI} :: \text{HB} : \text{HK},$$
$$\text{or } b-b' : b-b'-(y-b') :: a : x,$$
$$\text{or } b-b' : b-y :: a : x \ \therefore (b-b')x = a(b-y) \ldots\ldots 1$$

Again, by the problem, $\frac{b+y}{2} \times x = \text{S} \ \therefore (b+y)x = 2\text{S} \ldots 2$

Determining from equations 1, 2, the values of y and x,

$$y = \sqrt{\left\{b^2 - \frac{2\text{S}(b-b')}{a}\right\}}.$$

$$x = \frac{a}{b-b'}\left\{b - \sqrt{\left(b^2 - \frac{2\text{S}(b-b')}{a}\right)}\right\}.$$

Examples. 1st. Given a plank, 13 feet long, 18 in. broad at the greater end, and 14 in. broad at the smaller end: required to find the length which, taken at the greater end, will contain an area of 10 sq. ft.

In this instance, $b = 18$ in., $b' = 14$ in., $a = 13$ ft. $= 156$ in., S $= 10$ sq. ft. $= 1440$ sq. in.

CALCULATION OF y AND x.

$$y = \sqrt{\left(18^2 - \frac{2 \times 1440(18-14)}{156}\right)} = \sqrt{\left(324 - \frac{2 \times 1440 \times 4}{156}\right)} = \sqrt{\left(324 - \frac{960}{13}\right)}.$$

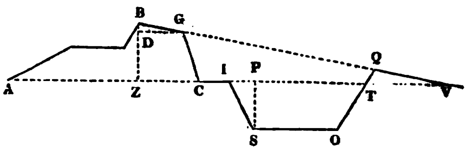
$= \sqrt{(324 - 73.846153)} = \sqrt{(250.153847)} = 15.816$ in. $=$ length of line of section.

$x = \frac{156}{18-14}(18 - 15.816) = \frac{156}{4} \times 2.184 = 39 \times 2.184 = 85.176$ in. $=$ length to be cut off.

2d. From the greater end of a trapezoidal field whose parallel ends and breadth measure 12, 8, and $10\frac{1}{2}$ chains respectively, it is required to cut off an area of six acres by a fence parallel to the parallel sides of the field: required the length of the fence and its distance from the greater side.

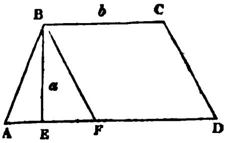
Ans. The length of the fence is 9.9139 chains, and its distance from the greater side 5.476 chains.

3d. Suppose the area of the profile ABGO $=100$ feet, and BD $=1$ foot; BZ $=10$ ft.; PS $=6$ ft. $=$ depth of ditch: DG $=6$ feet: ZP $=13$ feet; IP $=3$ ft. What must be the breadth of the ditch, so that its section ISOT shall be equal to the profile ABGC and TQV (the section of the glacis) to-



gether; when the superior slope BG and the exterior slope of the glacis QV, are in the same plane, the slopes IS, TO, being equal. Ans. IT, the breadth at top$=26.67$ feet.

80. In a trapezoid AC, given the shorter parallel side BC, the altitude BE, and the angles adjacent to the other parallel side AD, to find AD.



Let BC$=b$, BE$=a$, and the angles A, D, be given to find AD.

Draw BF parallel to CD: therefore, FD$=$BC$=b$, and the angle AFB$=$D.

$$\text{Then in the triangle ABE, AB} = \frac{\text{BE}}{\sin.\text{A}} = \frac{a}{\sin.\text{A}}, \text{ (Art. 56.)}$$

$$\text{And in the triangle ABF, AF} = \text{AB} \times \frac{\sin.(\text{A}+\text{D})}{\sin.\text{D}} \text{ (Art. 57.)}$$

$$\therefore \text{AF} = \frac{a}{\sin.\text{A}} \times \frac{\sin.(\text{A}+\text{D})}{\sin.\text{D}} = a \times \frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\sin.\text{D}},$$

$$\therefore \text{AF}+\text{FD or AD} = b + a \times \frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\sin.\text{D}}.$$

$$\text{If A}=\text{D}, \frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\sin.\text{D}} = \frac{2\sin.\text{A}\cos.\text{A}}{\sin.\text{A}\sin.\text{D}.} = 2\cot.\text{A}. \text{ (Art. 25. } b\text{)},$$

$$\text{and AD} = b + 2a\cot.\text{A}.$$

$$\text{If A}=\text{D}=45°, \cot.\text{A}=1, \text{ (Art. 33. } d.\text{)}$$

$$\text{And AD} = b + 2a.$$

Examples. 1st. Given $b=12$ ft.; $a=8$ ft.; A$=66° 18'$; D$=74° 52'$: required AD.

CALCULATION.

$$\text{A}+\text{D} = 66° 18' + 74° 52' = 141° 10',$$

$$\therefore \text{AD} = 12 + 8 \times \frac{\sin. 141° 10'}{\sin. 66° 18' \times \sin. 74° 52'}.$$

The log. of the second term of this value of AD is equal to

log. 8+log. sin. 141° 10′—10+arith. comp. of log. sin. 66° 18′+arith. comp. of log. sin. 74° 52′.

Now, log. 8 . . . . . . = 0.903090
log. sin. 141° 10′ . . . . = 9.797307
arith. comp. of log. sin. 66° 18′ = 0.038265
arith. comp. of log. sin. 74° 52′ = 0.015328
—10 . . . . . = —10.000000

The number of this log. is 5.6884 . . 0.754990

Therefore, the numerical value of the 2d term of the value of AD is 5.6884,

Therefore, AD$= 12 + 5.6884 = 17.6884$.

2d. The breadth of the bottom of a ditch is to be 18 feet, the depth 10 feet, and the slopes of the sides 75° and 80°: what must be the breadth of the excavation at top?

Ans. 22·4428 feet.

3d. In constructing a railroad the pathway of which is 24 feet broad, it is necessary to make a cutting 40 feet in depth: what must be the breadth of the cutting at top, 1st, supposing the slopes of the sides to be 65°, and 2d, supposing the slopes to be 45°?

Ans. 1st, 61.305 feet: 2d, 104 feet.

Since $\frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\sin.\text{D}}=\frac{\sin.\text{D}\cos.\text{A}+\cos.\text{D}\sin.\text{A}}{\sin.\text{A}\sin.\text{D}}=\frac{\cos.\text{A}}{\sin.\text{A}}+\frac{\cos.\text{D}}{\sin.\text{D}}=\cot.\text{A}+\cot.\text{D}$ (Art. 31. & 25. *b.*)

$$\therefore a\times\frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\ \sin.\text{D}}=\text{AF}=a(\cot.\text{A}+\cot.\text{D}).$$

This result may be obtained directly from the figure of this article:

$$\left.\begin{array}{l}\text{for AE}=\text{BE}\cot.\text{A (Art. 17.)}=a\cot.\text{A}\\ \quad\ \text{EF}=\text{BE}\cot.\text{D}\qquad\qquad\quad\ \ =a\cot.\text{D}\end{array}\right\}\therefore\text{AF}=a(\cot.\text{A}+\cot.\text{D}).$$

When a is a small number, the value of AD ($=b+\text{AF}$) may be more easily obtained from this expression, with the help of a table of natural tangents, than with logarithms from the formula $\text{AD}=b+a\times\frac{\sin.(\text{A}+\text{D})}{\sin.\text{A}\ \sin.\text{D}}$.

Taking, as an instance, the second example, in which $a=10$ ft.; $\text{A}=75°$; $\text{D}=80°$; $b=18$ ft.

$$\begin{array}{lr}\text{AD}=18+10(\cot.75°+\cot.80°).\quad\text{Now}\ \cot.75°= & \left.\begin{array}{r}.267949\\ \cot.80°=\underline{.176327}\end{array}\right\}\text{add}\\ \cot.75°+\cot.80°\quad.\quad.\quad.\quad= & \left.\begin{array}{r}.444276\\ \underline{10}\end{array}\right\}\text{multiply}\\ 10(\cot.75°+\cot.80°)\quad.\quad.\quad.\quad= & \left.\begin{array}{r}4.44276\\ \underline{18.00000}\end{array}\right\}\text{add}\\ \text{AD}=18+10(\cot.75°+\cot.80°)\quad.\quad.\quad.\quad= & 22.44276\end{array}$$

81. To find the area of a trapezium.

Given a diagonal of a trapezium and the perpendiculars let fall upon the diagonal from the opposite angles, to find the area of the trapezium.

Let the given diagonal be denoted by d, the given perpendiculars by a, a', and the area of the trapezium by S.

By the conditions of the problem, the trapezium is divided into two triangles which have for common base the diagonal d, and for altitudes the perpendiculars a, a', respectively.

Now the area of the triangle whose base is d and altitude $a=\frac{ad}{2}$ (Art. 7. *f.*)

And the area of the triangle whose base is d and altitude $a'=\frac{a'd}{2}$,

$$\therefore\ \text{S}=\frac{ad}{2}+\frac{a'd}{2}=\frac{(a+a')d}{2}.$$

Examples. 1st. Given $d = 25$ ft.; $a = 9\frac{1}{4}$ ft.; $a' = 13\frac{3}{4}$ ft.: required s.

$$\text{Ans. s} = \frac{(9.25 + 13.75) \times 25}{2} = \frac{23 \times 25}{2} = \frac{575}{2} = 287.5 \text{ sq. ft.}$$

2d. The diagonal of a trapezium is 34 ft. 9 in., and the perpendiculars 19 ft. 9 in., and 8 ft. 9 in.: required the area of the trapezium?

Ans. 495 sq. ft. 27 sq. in.

3d. The diagonal of a field, which has the form of a trapezium, is 1660 links, and the perpendiculars are 702 and 712 links: required the area of the field?

Ans. 11 acres, 2 roods, 37.792 poles.

a. Given all the sides of a trapezium and one of the angles to find the area of the trapezium.

Let the sides a, b, c, d, and the angle contained by the two sides a, b, be given; also let the required area be denoted by s; and let the angle contained by the two sides a, b, be denoted by the symbol (a, b).

Then, if a straight line is drawn to join the extremities of the sides a, b, the trapezium is divided into two triangles having this line for their common base.

In one of these triangles the sides a, b, and included angle (a, b) are given, and in the other, the sides c, d, are given. From the sides a, b, and angle (a, b) of the first triangle, the common base of both triangles (which may be denoted by d'), is found by Article 57. Case III. Then in the first triangle, the sides a, b, d', are given to find the area; and in the second, the sides c, d, d', are given to find the area.

In computing the area of a trapezium from the data of this problem, four distinct calculations are requisite:—

$$\left.\begin{array}{l}\text{The first, to find the angles } (a, d'), (b, d') \\ \text{The second, to find the length of the} \\ \quad \text{diagonal } d'\end{array}\right\}\text{, Art. 57. Case III.}$$

$$\left.\begin{array}{l}\text{The third, to find the area of the tri-} \\ \quad \text{angle whose sides are } a, b, d' \\ \text{The fourth, to find the area of the tri-} \\ \quad \text{angle whose sides are } c, d, d'\end{array}\right\}\text{, Art. 8. } a.$$

To escape this tedious calculation, it is usual to select other data: if the four sides are given, two opposite angles are also, generally, given. It is not, however, to be forgotten that the value of one of these angles, (c, d) for example, is determined by the values of a, b, c, d and (a, b); and that

d' being found and $\frac{c+d+d'}{2}$ being denoted by s, sin. $\frac{1}{2}(c, d)$, must be equal to $\sqrt{\left\{\frac{(s-c)(s-d)}{cd}\right\}}$, (Art. 57. Case IV.).

b. Given all the sides of a trapezium and two opposite angles to find the area of the trapezium. Let the sides a, b, c, d and the angles (a, b), (c, d) be given, to find the area s.

Considering the trapezium to be divided into two triangles in the first of which two sides a, b, and the included angle (a, b) are given, and in the second of which two sides c, d, and the included angle (c, d) are given; it follows from Article 76. *a*, that the areas of these triangles are $\frac{ab\sin.(a,b)}{2}$, $\frac{cd\sin.(c,d)}{2}$ respectively.

$$\text{Whence s} = \frac{ab\sin.(a,b)+cd\sin.(c,d)}{2}.$$

If $(a, b)+(c, d)=180°$, $\sin.(a, b)=\sin.(c, d)$. (Art. 18. *b.*),

$$\text{And s} = \frac{(ab+cd)\sin.(a, b)}{2}.$$

Examples. 1st. In a trapezium given $a=45$ ft.; $b=40$ ft.; $c=38$ ft.; $d=34$ ft.; $(a, b)=63° 28'$; $(c, d)=76° 56'$: required s.

Ans. s = 1434.484 sq. ft.

Calculation.

$$\text{Log.}\left(\frac{ab\sin.(a,b)}{2}\right)=\begin{cases}\text{log. } 45 \quad . \quad . \quad . \quad . & 1.653213\\ +\text{log. } 40 \quad . \quad . \quad . \quad . & +1.602060\\ +\text{log. sin. } 63° 28' \quad . \quad . & +9.951665\\ +\text{arith. comp. of log. } 2 \quad . & +9.698970\\ -20 \quad . \quad . \quad . \quad . \quad . & -20.000000\end{cases}$$

The number of this log. is 805.21 . . . . 2.905908

Therefore, the area of the triangle $a(a, b)b=805.21$ sq. ft.

$$\text{Log.}\left(\frac{cd\sin.(c,d)}{2}\right)=\begin{cases}\text{log. } 38 \quad . \quad . \quad . \quad . & 1.579784\\ +\text{log. } 34 \quad . \quad . \quad . \quad . & +1.531479\\ +\text{log. sin. } 76° 56' \quad . \quad . & +9.988607\\ +\text{arith. comp of log. } 2 \quad . & +9.698970\\ -20 \quad . \quad . \quad . \quad . \quad . & -20.000000\end{cases}$$

The number of this log. is 629.274 . . . . 2.798840

Therefore, the area of the triangle $c(c, d)d$ is 629.274 sq. ft.

Hence, the area of the trapezium = (805.21 + 629.274) = 1434.484 sq. ft.

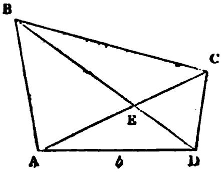
2d. The sides of a quadrilateral field are, $a=12.08$ chains; $b=8.56$ chains; $c=9.74$ chains; $d=14.25$ chains; also the angle $(a, b)=78° 40'$ and the angle $(c, d)=63° 55'$: required the area of the field.

Ans. 11 ac. 1 ro. 8.386 poles.

3d. The sides of a quadrilateral field are, $a=690$ yds.; $b=467$ yds.; $c=359$ yds.; $d=428$ yds.; also the angle $(a, b)=57° 30'$, and the angle $(c, d)=122° 30'$: required the area of the field.

Ans. 200677.2 sq. yds., or 41.46223 acres.

e. **Given one side of a trapezium, and the two angles subtended at each extremity of the given side by the two sides which are not adjacent to that extremity, to find the area of the trapezium.**



Given the base $\text{AD}=b$; the angles DAC and CAB, subtended at A by the sides DC, CB equal to A, A′ respectively; and the angles ADB, BDC subtended at D by the sides AB, BC equal to D, D′ respectively, to find S, the area of the trapezium ABCD.

Supposing AC and AB known;

$$\text{S}=\frac{\text{AD}\times\text{AC}\times\sin.\text{A}}{2}+\frac{\text{AC}\times\text{AB}\times\sin.\text{A}'}{2},\ (\text{Art. } 76.\ a,),$$

$$\text{Now, AC}=\frac{b\sin.(\text{D}+\text{D}')}{\sin.(\text{A}+\text{D}+\text{D}')},\ \text{and AB}=\frac{b\sin.\text{D}}{\sin.(\text{A}+\text{A}'+\text{D})}.$$

(Art. 57. Case I.)

$$\therefore\ \frac{\text{AD}\times\text{AC}\times\sin.\text{A}}{2}=\frac{b^2\sin.\text{A}\sin.(\text{D}+\text{D}')}{2\sin.(\text{A}+\text{D}+\text{D}')};\ \frac{\text{AC}\times\text{AB}\sin.\text{A}'}{2}=\frac{b^2\sin.\text{A}'\sin.\text{D}\sin.(\text{D}+\text{D}')}{2\sin.(\text{A}+\text{D}+\text{D}')\sin.(\text{A}+\text{A}'+\text{D})},$$

$$\text{and S}=\frac{b^2\sin.(\text{D}+\text{D}')\{\sin.\text{A}\sin.(\text{A}+\text{A}'+\text{D})+\sin.\text{A}'\sin.\text{D}.\}}{2\sin.(\text{A}+\text{D}+\text{D}')\sin.(\text{A}+\text{A}'+\text{D})}$$

It will be most convenient, in practice, to compute the areas of the triangles ACD, ACB, separately, and then to add them together.

Examples. 1st. Given $b=7.35$ chains; $\text{A}=25° 46'$; $\text{A}'=76° 18'$; $\text{D}=34° 25'$; $\text{D}'=62° 12'$: required S.

CALCULATION.

$\text{D}+\text{D}'=34° 25'+62° 12'=96° 37'$,
$\text{A}+\text{D}+\text{D}'=25° 46'+96° 37'=122° 23'$,
$\text{A}+\text{A}'+\text{D}=25° 46'+76° 18'+34° 25'=136° 29'$.

$$\text{Log.}\,(2\,\text{tr. ACD}) = \left\{\begin{array}{ll} 2\times\log.7.35 \;.\;.\;.\;.\;: & 1.732574 \\ +\log.\sin.25^\circ 46' \;.\;.\;.\;.\;. & +9.638197 \\ +\log.\sin.96^\circ 37' \;.\;.\;.\;.\;. & +9.997098 \\ +\text{arith. comp. of log. sin. } 122^\circ 23' \;. & +0.073409 \\ -20 \;.\;.\;.\;.\;.\;.\;. & -20.000000 \end{array}\right.$$

The number of this log. is 27.62344 . . . . 1.441278

Therefore, the area of the triangle ACD $=\frac{1}{2}$ of 27.62344 = 13.81172.

$$\text{Log.}\,(2\,\text{tr. ACB}) = \left\{\begin{array}{ll} 2\times\log.7.35 \;.\;.\;.\;.\;. & 1.732574 \\ +\log.\sin.76^\circ 18' \;.\;.\;.\;.\;. & +9.987465 \\ +\log.\sin.34^\circ 25' \;.\;.\;.\;.\;. & +9.752208 \\ +\log.\sin.96^\circ 37' \;.\;.\;.\;.\;. & +9.997098 \\ +\text{arith. comp. of log. sin. } 122^\circ 23' \;. & +0.073409 \\ +\text{arith. comp. of log. sin. } 136^\circ 29' \;. & +0.162055 \\ -30 \;.\;.\;.\;.\;.\;.\;. & -30.000000 \end{array}\right.$$

The number of this log. is 50.67674 . . . . 1.704809

Therefore, the area of the triangle ACB $=\frac{1}{2}$ of 50.67674 = 25.33837.

Whence S = 13.81172 + 25.33837 = 39.15009 sq. chains = 3.915009 acres.

2d. Given $b = 311$ yds.; $\text{A} = 41^\circ\ 19'$; $\text{A}' = 44^\circ\ 20'$; $\text{D} = 24^\circ\ 10'$; $\text{D}' = 37^\circ\ 4'$: required S.

Ans. S = 8.5022 acres.

d. Given the diagonals of a trapezium and the angle which they form with each other, to find the area of the trapezium.

Given $\text{AC} = d$; $\text{BD} = d'$, and the angle AEB or DEC = E, (see the last figure) to find S, the area of the trapezium ABCD.

Since AEB + BEC = 180°; sin. AEB = sin. BEC = sin. E. (Art. 18. *b.*)

And since DEC + AED = 180°; sin. DEC = sin. AED = sin. E.

$$\text{Now, area ABE} = \frac{\text{AE}\times\text{EB}\times\sin.\text{AEB}}{2} = \frac{\text{AE}\times\text{EB}\times\sin.\text{E}}{2}$$

$$\text{And, area BEC} = \frac{\text{EC}\times\text{EB}\times\sin.\text{BEC}}{2} = \frac{\text{EC}\times\text{EB}\times\sin.\text{E}}{2},$$

$$\therefore\ \text{area ABC} = \frac{(\text{AE}+\text{EC})\,\text{EB}\sin.\text{E}}{2} = \frac{\text{AC}\times\text{EB}\sin.\text{E}}{2},$$

$$\text{Also, area AED} = \frac{\text{AE}\times\text{ED}\sin.\text{AED}}{2} = \frac{\text{AE}\times\text{ED}\sin.\text{E}}{2},$$

$$\text{And, area CED} = \frac{\text{EC}\times\text{ED}\sin.\text{DEC}}{2} = \frac{\text{EC}\times\text{ED}\sin.\text{E}}{2},$$

$$\therefore\ \text{area ADC} = \frac{(\text{AE}+\text{EC})\,\text{ED}\sin.\text{E}}{2} = \frac{\text{AC}\times\text{ED}\sin.\text{E}}{2},$$

$$\therefore \text{ABC} + \text{ADC} = \frac{\text{AC} \times \text{EB} \sin. \text{E}}{2} + \frac{\text{AC} \times \text{ED} \sin. \text{E}}{2} = \frac{\text{AC}(\text{EB} + \text{ED}) \sin. \text{E}}{2} = \frac{\text{AC} \times \text{BD} \sin. \text{E}}{2}$$

$$= \frac{d \times d' \sin. \text{E}}{2} = \text{S}.$$

It is evident that $\frac{\text{AC} \times \text{BD} \sin. \text{E}}{2}$ or $\frac{d \times d' \sin. \text{E}}{2}$ expresses the area of a triangle of which two sides and the included angle are d, d', E, respectively, (Art. 76. *a.*) Consequently a trapezium is equivalent to a triangle, two of whose sides and the angle included by them are equal to the diagonals of the trapezium and the angle which they form with each other.

Examples. 1st. Given $d = 20$ chains; $d' = 26$ chains; $\text{E} = 56°$: required S.

Ans. $\text{S} = 21.56$ acres.

2d. Given $d = 224$ yds.; $d' = 188$ yds.; $\text{E} = 40° 5'$: required S.

Ans. $\text{S} = 13558$ sq. yds.

3d. Given $d = 15.24$ yds.; $d' = 11.14$ yds.; $\text{E} = 105° 28'$: required S.

Ans. $\text{S} = 81.813$ sq. yds.

82. To find the area of an irregular polygon.

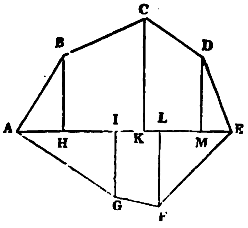
A diagonal being drawn to join the two extreme angles of the polygon, and perpendiculars being let fall from all the other angles upon this diagonal; let there be given the distance from one extremity of the diagonal to the foot of each perpendicular, the length of the diagonal, and the length of each perpendicular, to find the area of the polygon.

The polygon is divided, by the diagonal and perpendiculars, into triangles whose bases and altitudes are known, and into trapezoids whose parallel sides and the distances between them are known: the areas of these triangles and trapezoids are found by Articles 76. and 77.; and their sum is equal to the area of the polygon.

Examples.

1st. Given AH = 20 yds.; BH = 35 yds.
AI = 42 . . IG = 28 . .
AK = 56 . . KC = 50 . .
AL = 59 . . LF = 36 . .
AM = 78 . . MD = 32 . .
AE = 94 . .

to find the area of the irregular polygon ABCDEFG.



Calculation.

$\text{AH}=20:\text{BH}=35 \therefore \frac{\text{AH}\times\text{BH}}{2}=\frac{20\times35}{2}=350$ sq. yds. = area of triangle ABH.

$\text{HK}=\text{AK}-\text{AH}=56-20=36: \frac{\text{BH}+\text{CK}}{2}=\frac{35+50}{2}=\frac{85}{2}=42.5,$

$\therefore \frac{\text{BH}+\text{CK}}{2}\times\text{HK}=42.5\times36$. $=1530$ sq. yds. = area of trapezoid HC.

$\text{KM}=\text{AM}-\text{AK}=78-56=22: \frac{\text{CK}+\text{DM}}{2}=\frac{50+32}{2}=\frac{82}{2}=41.$

$\therefore \frac{\text{CK}+\text{DM}}{2}\times\text{KM}=41\times22$. $=902$ sq. yds. = area of trapezoid KD.

$\text{ME}=\text{AE}-\text{AM}=94-78=16: \text{DM}=32,$

$\therefore \frac{\text{ME}\times\text{DM}}{2}=\frac{16\times32}{2}$. $=256$ sq. yds. = area of triangle DME.

$\text{AI}=42:\text{IG}=28 \therefore \frac{\text{AI}\times\text{IG}}{2}=\frac{42\times28}{2}=588$ sq. yds. = area of triangle AIG.

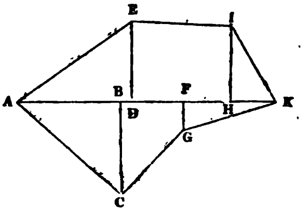
$\text{IL}=\text{AL}-\text{AI}=59-42=17: \frac{\text{IG}+\text{LF}}{2}=\frac{28+36}{2}=\frac{64}{2}=32.$

$\therefore \frac{\text{IG}+\text{LF}}{2}\times\text{IL}=32\times17$. $=544$ sq. yds. = area of trapezoid IF.

$\text{LE}=\text{AE}-\text{AL}=94-59=35:\text{LF}=36.$

$\therefore \frac{\text{LE}\times\text{LF}}{2}=\frac{35\times36}{2}$. $=630$ sq. yds. = area of triangle LEF.

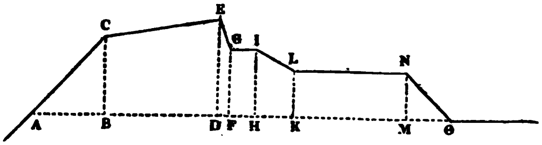
The sum of the areas of these triangles and trapezoids - - } = 4800 sq. yds. = area of polygon ABCDEFG.



2. Given AB = 450 links; BC = 380 links;
AD = 500 DE = 325;
AF = 720 FG = 114;
AH = 916 HI = 300;
AK = 1135. Required the area AEIKGC.

Ans. Area AEIKGC = 4.19945 ac.

3. Given the following measurement of the profile of part of a work: viz.



AB = 12 feet; BC = 12 feet;
AD = 30 .. DE = 15 ..
AF = $31\frac{1}{2}$ .. FG = $10\frac{1}{2}$ ..
AH = $35\frac{1}{2}$ .. HI = $10\frac{1}{2}$ ..
AK = $41\frac{1}{2}$ .. KL = $7\frac{1}{2}$ ..
AM = $59\frac{1}{2}$ .. MN = 7 ..
AO = $66\frac{1}{2}$. Required the area of the profile.

Ans. Area of the profile = 585.125 square feet.

a. Diagonals being drawn dividing an irregular polygon into triangles, and perpendiculars being let fall upon these diagonals from the opposite angles, let there be given the lengths of the diagonals and also of the perpendiculars, to find the area of the polygon.

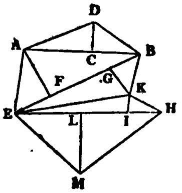
By the conditions of the problem, the polygon is divided into triangles whose bases and altitudes are given. The areas of the several triangles can consequently be computed by article 76.: and the sum of these areas is the area of the polygon.

If the diagonals and perpendiculars are drawn in such a manner that the triangles, taken two by two, have a common base, the length of the calculation is somewhat abridged.

Example. 1st. In the irregular polygon ADBKHME, given

diagonal AB = 24.8 yds.; perpendicular CD = 4 yds:
diagonal EB = 29 yds.; perp. AF = 11 yds.; perp. GK = 6.6 yds:
diagonal BH = 30.5 yds.; perp. KI = 6 yds.; perp. ML = 11.2 yds.

Required the area of the polygon.



CALCULATION.

AB = 24.8; CD = 4 ∴ $\frac{AB \times CD}{2} = \frac{24.8 \times 4}{2}$ . . . sq. yds. = 49.6 = area of triangle ADB.

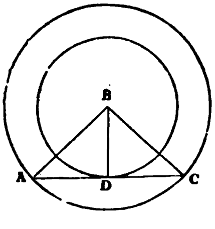
EB = 29; AF + GK = 11 + 6.6 = 17.6 ∴ $\frac{ED(AF+GK)}{2} = \frac{29 \times 17.6}{2}$ = 255.2 = area ABKE.

EH = 30.5; KI + ML = 6 + 11.2 = 17.2 ∴ $\frac{EH(KI+ML)}{2} = \frac{30.5 \times 17.2}{2}$ = 262.3 = area EKHM.

The sum of these areas (or the area of the proposed polygon) = 567.1 sq. yds.

83. To find the area of a regular polygon.

Given the number of sides of a regular polygon and the radius of the circumscribed circle, to find the area of the polygon.



Let n be the number of sides of a regular polygon; BA = r, the radius of the circumscribed circle; and S the required area. Then, since the polygon has n sides, each equal to AC, and by consequence n angles each equal to ABC, the angle ABC $= \frac{360^\circ}{n}$,

$$\therefore \text{ area ABC or } \frac{\text{AB} \times \text{BC sin. ABC}}{2} = \tfrac{1}{2} r^2 \text{ sin. } \frac{360^\circ}{n},$$

(Art. 76. *a.*)

$$\text{And S, or area ABC} \times n = \tfrac{1}{2} n r^2 \text{ sin. } \frac{360^\circ}{n}.$$

Hence

$$\log.\text{S} = \log. n + 2 \log. r + \log.\text{sin.} \frac{360^\circ}{n} + \text{arith.comp.of} \log. 2 - 20.$$

Examples. 1st. Given $n = 9$; $r = 5.8$: required S. *Ans.* S = 97.3051.

CALCULATION.

$$\text{Log. S} = \begin{cases} \log. 9 \ \ldots\ldots & 0.954243 \\ + 2 \log. 5.8 \ \ldots\ldots & + 1.526856 \\ + \log. \sin. 40^\circ \left(= \frac{360^\circ}{9}\right) \ \ldots & + 9.808067 \\ + \text{arith. comp. of log. } 2 \ \ldots & + 9.698970 \\ - 20 \ \ldots\ldots & - 20.000000 \end{cases}$$

The number of this log. is 97.3051 . . . 1.988136

Whence the area of the proposed polygon is 97.3051.

2d. Given $n = 5$; $r = 40$ feet: required S.
Ans. S = 3804.2193 sq. feet.

3d. Given $n = 90$; $r = 40$ feet: required S.
Ans. S = 5022.4768 sq. ft.

a. Given the number of sides of a regular polygon and the radius of the inscribed circle to find the area of the polygon.

Let n be the number of sides of a regular polygon; BD = r', the radius of the inscribed circle; and S the required area.

$$\text{The area of the triangle ABC} = \frac{\text{AC}}{2} \times \text{BD, (Art. 76.)} = \text{AD} \times r',$$

$$\text{Also, } \frac{\text{AD}}{r'} = \text{tan. ABD (Art. 17.)} = \text{tan.} \frac{\text{ABC}}{2} = \text{tan.} \frac{360^\circ}{2n} = \text{tan.} \frac{180^\circ}{n}.$$

$$\therefore \text{ AD} = r' \text{ tan.} \frac{180^\circ}{n},$$

$$\text{and AD} \times r' = r'^2 \text{ tan.} \frac{180^\circ}{n},$$

$$\therefore n \times \text{AD} \times r' \text{ or S} = n r'^2 \text{ tan.} \frac{180^\circ}{n}.$$

$$\text{And log. S} = \log. n + 2 \log. r' + \log. \tan. \frac{180^\circ}{n} - 10.$$

Examples. 1st. Given $n=15$; $r'=9$: required s. *Ans.* s$=258.257$.

CALCULATION.

$$\text{Log. s}=\begin{cases}\text{log. }15 \ \ldots\ldots & 1.176091\\ +2\,\text{log. }9 \ \ldots\ldots & +1.908486\\ +\text{log. tan. }12^{\circ}\left(=\dfrac{180^{\circ}}{15}\right) \ \ldots & +9.327475\\ -10 \ \ldots\ldots & -10.000000\end{cases}$$

The number of this log. is 258.257 . . . . 2.412052

$\therefore$ s$=258.257$.

2d. Given $n=5$; $r'=17.2$ yds.: required s.
Ans. s$=1074.7$ sq. yds.

3d. Given $n=25$; $r'=1$ inch: required s.
Ans. s$=3.15823$ sq. inches.

b. Given the number of sides of a regular polygon and the length of a side, to find the area of the polygon.

Let n be the number of sides of a regular polygon, l, the length of a side, and s, the required area.

The area of the triangle $\text{ABC}=\dfrac{\text{AC}}{2}\times\text{BD}=\dfrac{l}{2}\times\text{BD}$,

Also $\text{BD}=\text{AD}\cot.\ \text{ABD}=\dfrac{l}{2}\cot.\ \dfrac{180^{\circ}}{n}$, (Art. 17.)

$$\therefore\ \frac{\text{AC}}{2}\times\text{BD}=\frac{l}{2}\times\frac{l}{2}\cot.\ \frac{180^{\circ}}{n}=\frac{l^2}{4}\cot.\ \frac{180^{\circ}}{n},$$

$$\therefore\ \text{s}=\frac{nl^2}{4}\cot.\ \frac{180^{\circ}}{n},$$

And $\log.\ \text{s}=\log.\ n+2\log.\ l+\log.\ \cot.\ \dfrac{180^{\circ}}{n}+\text{arith. comp. of }\log.\ 4-20.$

Examples. 1st. Given $n=10$; $l=20$ feet: required s.
Ans. s$=3077.6836$ sq. ft.

2d. Given $n=7$; $l=19.38$ yds.: required s.
Ans. s$=1364.84135$ sq. yds.

3d. Given $n=3$; $l=20$ ft.: required s.
Ans. s$=173.20506$ sq. ft.

4th. Given $n=6$; $l=20$ ft.: required s.
Ans. s$=1039.23048$ sq. ft.

84. To find the length of the circumference of a circle.

Given the radius of a circle to find the length of the circumference of the circle.

Let r be the radius of the circle, and let the length of the circumference be denoted by c;

$$\text{then, by Art. 8. } b.,\ c=2\pi r.$$

If d, the diameter, is given; since $d=2r$, $c=\pi d$.

Hence, to find the length of the circumference of a circle when the radius or the diameter is given,

Multiply twice the radius by 3.1416: or

Multiply the diameter by 3.1416: the result expresses the length of the circumference.

Examples. 1st. Given $r=25$ feet: required c.
Ans. $c=157.08$ feet.

CALCULATION.

$$r=25 \quad \therefore\ 2\pi r=2\times 3.1416\times 25=3.1416\times 50=157.08.$$

2d. Given $d=7$: required c.
Ans. $c=22$ nearly.

3d. Given $d=113$: required c.
Ans. $c=355$ nearly.

4th. Given $r=3963.7$ miles: required c.
Ans. $c=24904.71984$ miles.

a. Given the length of the circumference of a circle, to find the radius or the diameter.

Let c be the length of the circumference; and let the radius and diameter be denoted by r, d, respectively.

$$\text{Then since } 2\pi r=c,\ r=\frac{c}{2\pi},$$

$$\therefore\ 2r \text{ or } d=\frac{c}{\pi}.$$

Consequently, to find the radius of a circle when the circumference is given,

Divide the circumference by 2π; the result is the radius:

And to find the diameter when the circumference is given,

Divide the circumference by π; the result is the diameter.

Examples. 1st. Given $c=157.08$ ft.: required r.
Ans. $r=25$ feet.

2d. Given $c=12$ yds.: required d.
Ans. $d=3.819708$ yds.

85. Of the three quantities; 1., the length of an arc of a circle; 2., the number of degrees contained in the arc; 3., the radius of the circle; given any two to determine the third.

If l denote the length of the arc; m, the number of degrees in the arc; and r, the radius of the circle, it is proved in Art. 14. *c.*, that

$$l=2\pi r\times\frac{m}{360^\circ} \quad . \;.\;.\;.\;.\;.\;. \quad 1$$

$$m=\frac{l}{2\pi r}\times 360^\circ \quad . \;.\;.\;.\;.\;.\;. \quad 2$$

$$r=\frac{l}{2\pi}\times\frac{360^\circ}{m} \quad . \;.\;.\;.\;.\;.\;. \quad 3$$

Consequently,
if r and m are given, l is determined by formula 1:
if l and r are given, m is determined by formula 2:
and, if l and m are given, r is determined by formula 3.

Examples. 1st. Given $r=11.3$; $m=52^\circ\,15'$: required l.
Ans. $l=10.304834$.

Calculation.

$$52^\circ\,15'=52^\circ\tfrac{1}{4} \text{ and } 52^\circ\tfrac{1}{4}\div 360^\circ=\frac{209}{1440},$$

$$\therefore l=2\times 3.1416\times 11.3\times\frac{209}{1440},$$

$$\text{And log. } l=\left\{\begin{array}{lr} \text{log. } 2 \;.\;.\;.\;.\;.\;.\;. & 0.301030 \\ +\text{log. } \pi \;.\;.\;.\;.\;.\;. & +0.497150 \\ +\text{log. } 11.3 \;.\;.\;.\;.\;. & +1.053078 \\ +\text{log. } 209 \;.\;.\;.\;.\;. & +2.320146 \\ +\text{arith. comp. of log. } 1440 \;. & +6.841638 \\ -10 \;.\;.\;.\;.\;.\;. & -10.000000 \end{array}\right.$$

The number of this log. is 10.304834 . . 1.013042

$$\therefore l=10.304834.$$

2d. Given $r=9$; $m=38^\circ.9424412$: required l.
Ans. $l=6.11707$.

3d. Given $l=41\frac{1}{3}$; $r=19.9855$: required m.
Ans. $m=118^\circ\,29'\,48''$.

4th. Given $l=6.2832$; $m=30^\circ$: required r.
Ans. $r=12$.

86. To find the area of a circle.

Given the diameter of a circle to find the area.

Let $2r$ or d be the diameter; and let the required area be denoted by S: then,

1st. If the diameter of a circle is 1, the circumference $=\pi$, (Art. 8. *a.*);

Whence, since the area of a circle is obtained by multiplying half the circumference of the circle by half the diameter, (Circle, Prop. 8.) the area of the circle whose diameter is $1=\frac{1}{2}$ of $1\times\frac{1}{2}$ of $\pi=\frac{\pi}{4}=\frac{3.1415926}{4}=.7853981=$ $.7854$ nearly.

2d. Since the areas of circles are proportional to the squares of the diameters of the circles, (Circle, Prop. 9.); and since 1, d, are the diameters of two circles, and $\frac{\pi}{4}$, S, their respective areas,

$$\therefore\ 1^2 : d^2 :: \frac{\pi}{4} : \text{S},\quad \therefore\ \text{S}=\frac{\pi}{4}d^2,$$

or, since $d=2r$, $d^2=4r^2$: whence, by substitution,

$$1 : 4r^2 :: \frac{\pi}{4} : \text{S},\quad \therefore\ \text{S}=\pi r^2.$$

Consequently, to find the area of a circle,

Multiply the square of the diameter of the circle by .7854: or

Multiply the square of the radius of the circle by 3.1416: the product expresses the area of the circle.

The equation $\text{S}=\pi r^2$ may be put under the form $\text{S}=r\times\frac{2\pi r}{2}$. Now, in this expression, r being the radius of a circle, $2\pi r$ is the circumference of that circle. But if r is the altitude, and $2\pi r$, the base of a triangle, $r\times\frac{2\pi r}{2}$ expresses the area of the triangle. Whence the area of a circle is equal to that of a triangle whose base is equal to the circumference, and its altitude to the radius of the circle.

Examples. 1st. Given $d=12$: required S.
Ans. $\text{S}=113.0976$.

CALCULATION.

$$\text{S}=12^2\times .7854=144\times .7854=113.0976.$$

2d. Given $r = 100$ feet: required s.
Ans. $s = 31415.9$ sq. feet.

3d. Given $r = 27\frac{3}{4}$ yds.: required s.
Ans. $s = 2419.22835$ sq. yds.

a. Given the area of a circle to find the radius.

$$\text{Since } \pi r^2 = s;\ r^2 = \frac{s}{\pi},\ \text{and } r = \sqrt{\frac{s}{\pi}}.$$

r, being determined, $2r$ or d, and $2\pi r$ or c can be obtained from r.

Example. Let $s = 4840$ sq. yds.: required r. *Ans.* $r = 39.25063$ yds.

CALCULATION.

$$r = \sqrt{\frac{4840}{3.1416}}.$$

$$\therefore \log. r = \tfrac{1}{2}\left\{\begin{array}{ll} \log. 4840 \quad . \quad . \quad . \quad . & 3.684845 \\ +\text{arith. comp. of log. } \pi \quad . \quad . & +9.502850 \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{array}\right\}$$

$$2)\underline{3.187695}$$

The number of this log. is 39.25063 . . 1.593487

$\therefore r = 39.25063$ yds.

87. To find the area of a sector of a circle.

Given the length of the radius and the number of degrees in the arc of a sector of a circle to find the area of the sector.

Let r be the radius; m, the number of degrees in the arc; and s, the required area.

Then πr^2 = the area of the whole circle whose radius is r (Art. 86.),

$$\therefore 360^\circ : m :: \pi r^2 : s, \text{ (Euc. VI. 33.)},$$

$$\therefore s = \pi r^2 \times \frac{m}{360^\circ}.$$

Since $\pi r^2 \times \frac{m}{360^\circ} = r \times \frac{1}{2}$ of $2\pi r \times \frac{m}{360^\circ}$; and $2\pi r \frac{m}{360^\circ}$ expresses the length of an arc of m° in the circle whose radius is r, it follows that the area of a sector of a circle is obtained by multiplying half the arc of the sector by the radius of the circle.

Hence, also, the area of a sector of a circle is equal to the area of a triangle whose base is equal to the length of the arc of the sector, and its altitude equal to the radius of the sector.

Examples. 1st. Given $r=16$; $m=120°$: required s.
Ans. s$=268.0827$.

CALCULATION.

$$\frac{m}{360°}=\frac{120°}{360°}=\frac{1}{3}\;\therefore\;\text{s}=\pi\times 16^2\times\tfrac{1}{3}:$$

$$\text{And log. s}=\begin{cases}\text{log. }\pi\;.\;.\;.\;.\;.\;.\;. & 0.497150\\ +2\times\text{log. }16\;.\;.\;.\;. & +2.408240\\ +\text{arith. comp. of log. }3\;.\;. & +9.522879\\ -10\;.\;.\;.\;.\;.\;.\;. & -10.000000\end{cases}$$

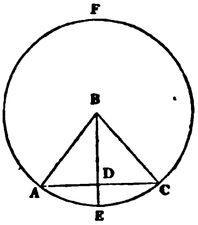
The number of this log. is 268.0827 . . . 2.428269

$$\therefore\text{s}=268.0827.$$

2d. Given $r=16$; $m=240°$: required s.
Ans. s$=536.1664$.

3d. Given $r=75.6$ feet; $m=7°\,30'$: required s.
Ans. s$=374.07$ sq. feet.

a. Given the radius and the chord of the arc of a sector to find the area of the sector.



Let AB$=r$, be the radius; AC$=c$, the chord of the arc, and s, the required area.

Since the arc AEC and the angle ABC have the same measure (Art. 14. *a.*); $\pi r^2\times\frac{\text{AEC}}{360°}=\pi r^2\times\frac{\text{ABC}}{360°}$: if therefore the angle ABC were known, s could be determined.

To find the angle ABC.

Let BD be drawn perpendicular to AC. Then AD$=\frac{c}{2}$;

$<\text{ABD}=\frac{\text{ABC}}{2}$; and sin. ABD$=\frac{\text{AD}}{\text{AB}}$ (Art. 17) $=\frac{c}{2r}$,

$\therefore$ sin. $\frac{\text{ABC}}{2}=\frac{c}{2r}$. Whence the angle ABC is determined, and consequently also s.

The chord AC may belong to either the sector ABCE or the sector ABCF: if the area of ABCE is required, it is necessary, in the expression $\pi r^2\times\frac{m}{360°}$, to substitute for m the value of the angle ABC; and, if the area of ABCF, to substitute for m the value of the angle $360°-$ABC.

Examples. 1st. Given $r=16$; $c=12$: required S. *Ans.* S $=98.409$.

CALCULATION OF THE ANGLE ABC.

$$\frac{c}{2r}=\frac{12}{32}=\frac{3}{8}=.375.$$

$$\therefore \text{log. sin. } \frac{\text{ABC}}{2}=\begin{cases} \text{log. } .375 \quad . \quad . \quad . \quad . \quad . & \bar{1}.574031 \\ +10 \quad . \quad . \quad . \quad . \quad . \quad . & +10.000000 \end{cases}$$

$$\text{The angle of this log. sin. is } 22^\circ\, 1'\, 30'' \quad . \quad . \quad 9.574031$$

$$\therefore \frac{\text{ABC}}{2}=22^\circ\, 1'\, 30'' \text{ and ABC}=44^\circ\, 3'=m.$$

CALCULATION OF S.

$$\frac{m}{360^\circ}=\frac{44^\circ\, 3'}{360^\circ}=\frac{2643}{21600}=\frac{881}{7200},$$

$$\therefore \text{S}=\pi \times 16^2 \times \frac{881}{7200}.$$

$$\text{And log. S}=\begin{cases} \text{log. } \pi \quad . \quad . \quad . \quad . \quad . \quad . & 0.497150 \\ +2 \times \text{log. } 16 \quad . \quad . \quad . \quad . & +2.408240 \\ +\text{log. } 881 \quad . \quad . \quad . \quad . \quad . & +2.944976 \\ +\text{arith. comp. of log. } 7200 \quad . & +6.142668 \\ -10 \quad . \quad . \quad . \quad . \quad . \quad . & -10.000000 \end{cases}$$

$$\text{The number of this log. is } 98.409 \quad . \quad . \quad . \quad 1.993034$$

$$\therefore \text{S}=98.409.$$

2d. Given $r=9$, $c=6$: required S.
Ans. S $=27.52664$.

3d. Given $r=27.4358$; $c=50.8$: required S.
Ans. S $=890.58$.

b. Given the chord of the arc of a sector and the perpendicular from the centre of the circle upon the chord, to find the area of the sector.

Let AC $=c$ be the chord, and BD $=p$, the perpendicular.

Then, since $\text{AB}=\sqrt{\text{AD}^2+\text{BD}^2}$,

$$r=\sqrt{\frac{c^2}{4}+p^2}.$$

r can therefore be determined from c and p: this case is thus reduced to Case *a*.

c. Given the chord of the arc of a sector and the versed sine of half the arc to find the area of the sector.

Let AC $=c$ be the chord, and DE $=v$ the versed sine of $\frac{\text{AEC}}{2}$ or $\frac{\text{ABC}}{2}$.

Since $\text{AD}^2 = \text{ED}(2r - \text{ED})$, (Euc. III. 35.)

$$\text{or } \frac{c^2}{4} = v(2r - v) = 2vr - v^2,$$

$$\therefore r = \frac{c^2 + 4v^2}{8v}.$$

Consequently r can be determined from c, v: and this case, also, is reduced to Case *a.*

d. Given the chord of the arc of a sector and the angle at the centre, to find the area of the sector.

Let c be the chord; and since the angle at the centre and the arc of the sector have the same measure, let m express the value of either.

$$\text{Then, since by Case } a,\ \frac{c}{2r} = \sin.\tfrac{1}{2}\,\text{ABC} = \sin.\tfrac{1}{2}m,$$

$$\therefore r = \frac{c}{2\sin.\frac{1}{2}m}, \text{ and } r^2 = \frac{c^2}{4\sin.^2\frac{1}{2}m}.$$

Substituting this value of r^2 in the formula $\text{S} = \pi r^2 \times \frac{m}{360^\circ}$.

$$\text{S} = \frac{\pi c^2}{4\sin.^2\frac{1}{2}m} \times \frac{m}{360^\circ}.$$

Ex. Given $c = 8$; $m = 24^\circ$: required S. *Ans.* $\text{S} = 77.5213$.

CALCULATION.

$$\frac{m}{360^\circ} = \frac{24^\circ}{360^\circ} = \frac{1}{15}.\ \frac{1}{2}m = 12^\circ,$$

$$\therefore \text{S} = \frac{\pi \times 8^2}{4\sin.^2 12^\circ} \times \frac{1}{15}.$$

$$\text{And log. S} = \begin{cases} \text{log. } \pi \ \ldots\ldots & 0.497150 \\ +2\,\text{log. } 8 \ \ldots\ldots & +1.806180 \\ +\text{arith. comp. of log. } 4 \ \ldots & +9.397940 \\ -2 \times \text{log. sin. } 12^\circ + 20 \ \ldots & +1.364242 \\ +\text{arith. comp. of log. } 15 \ \ldots & +8.823909 \\ -20 \ \ldots\ldots & -20.000000 \end{cases}$$

The number of this log. is 77.5213 . . 1.889421

$$\therefore \text{S} = 77.5213.$$

88. To find the area of a segment of a circle.

Given the radius and the number of degrees in the arc of a segment;

or the radius and the chord of a segment;

or the chord and the altitude of a segment;

or the chord and the number of degrees in the arc of a segment; to find the area of the segment.

The area of the segment AECD is equal to the excess of the area of the sector ABCE (see the fig. of Art. 87. *a.*), over the area of the triangle ABC. Consequently, if from the area of the sector (found by Art. 87.), the area of the triangle (found by Art. 76.) is subtracted, the remainder is the area of the segment.

If the segment is greater than a semicircle, as AFC, it will be necessary to add the area of the triangle ABC to the area of the sector AFCB, for the area of the segment.

Examples. 1st. Given $r=16$; $m=120°$: required S, the area of the segment.

The area of the sector (see Ex. 1. Art. 87.), is 268.0827.

And the area of the triangle is

$$\frac{r^2 \sin. m}{2} \text{ (Art. 76. } a) = \frac{16^2 \sin. 120°}{2} = 110.8054,$$

$$\therefore \text{S} = 268.0827 - 110.8054 = 157.2773 = \text{the area required.}$$

2d. Given $r=16$; $c=12$: required the area of the segment.

The area of the sector (see Ex. 1. Art. 87. *a*) is 98.409.

$$\text{The area of the triangle} = \sqrt{r^2 - \frac{c^2}{4}} \times \frac{c}{2} = \sqrt{220} \times 6 = 14.8324 \times 6 = 88.9944,$$

$$\therefore \text{S} = 98.409 - 88.9944 = 9.4146.$$

3d. Given the chord c, and the altitude v, of a segment, to find the area of the segment.

The area of the sector is found by Art. 87. *c.*

And since v, the altitude of the segment, is equal to the versed sine of half the arc of the segment,

$$r = \frac{c^2 + 4v}{8v} \text{ (Art. 87. } c),$$

$$\therefore r - v = \frac{c^2 + 4v^2}{8v} - v = \frac{c^2 - 4v^2}{8v}.$$

Now c is the base, and $r-v$ the altitude, of the triangle ABC; whence the area of the triangle $\text{ABC} = \frac{(r-v)c}{2}$, can be found. (Art. 76.)

Ex. Given $c=23$; $v=3.5$; required S.

Ans. S $=54.584$.

4. Given c, the chord, and m, the number of degrees in the arc of the segment, to find the area of the segment.

The area of the sector is found by Article 87. *d.*

To find the area of the triangle ABC,

$$\text{AD} = \frac{c}{2}; \quad < \text{ABD} = \frac{m}{2}.$$

$$\frac{\text{BD}}{\text{AD}} = \text{cot. ABD (Art. 17.)}, \therefore \text{BD} = \text{AD cot. ABD} = \frac{c}{2} \text{cot. } \tfrac{1}{2} m.$$

Wherefore BD is determined, and consequently also $\frac{\text{AC} \times \text{BD}}{2}$, or the area of the triangle ABC.

Ex. Given $c=8$; $m=24°$: required the area of the segment.

The area of the sector (see Art. 87. *d.*) is 77.5213,

the area of the triangle $= \frac{8 \times 4 \text{ cot. } 12°}{2}$ is 75.2739.

$\therefore$ the area of the segment is 2.2274.

Other examples. 1st. Given $c=12$; $r=10$: required S.
Ans. S $=16.3504$.

2d. Given $r=40$; $v=50$; required S.
Ans. S $=3304.55$.

SECTION IX.

MENSURATION OF SOLIDS.

89. The plane figures which bound a solid are termed faces of the solid; and the straight lines formed by the intersection of two faces which cut each other are termed edges of the solid.

a. Of the two opposite parallel faces of a prism or cylinder (Prisms, &c., Def. 2.), that on which the solid is placed is the base and the opposite face is the top of the prism or cylinder. The face on which a pyramid or cone is placed is the base, and the opposite angle the vertex of the pyramid or cone.

b. The other plane faces of prisms and pyramids are termed lateral faces.

The other surface of a cylinder or cone is termed a convex surface; and the same name is employed to describe the surface of a sphere.

c. If the base and top of a prism are regular polygons, the straight line joining the centres of these polygons is termed the axis of the prism.

d. If a pyramid or cone is cut by a plane parallel to the base, the portion of the solid comprehended between the section and the base is termed a frustum of a pyramid or cone, or a truncated pyramid or cone.

e. If a prism or cylinder is cut by a plane inclined to the base, the portion of the solid comprehended between the section and the base is termed a frustum of a prism or cylinder, or a truncated prism or cylinder.

The frustum of a triangular prism is sometimes termed a wedge.

f. The area of the surface of any solid bounded by plane faces is equal to the sum of the areas of all these faces. Thus the area of the surface of a prism is equal to the sum of the areas of the parallelograms which form the lateral faces of the prism, plus the areas of the two equal figures which form the base and top of the prism: the area of the surface of a pyramid is equal to the sum of the areas of the triangles which form the lateral faces of the pyramid, plus the area of the figure which forms the base of the pyramid. The necessary data being given, these areas are found by the rules for the calculation of the areas of plane rectilineal figures in Section VIII.

The areas of the surfaces of prisms and upright pyramids, whose bases are regular polygons, can, however, be found without computing separately the area of every face.

90. To find the area of the lateral surface of an upright prism.

Given the perimeter of the base and the altitude (or length of a lateral edge) of an upright prism to find the area of the lateral surface of the prism.

Let p be the perimeter of the base; h the altitude of the prism, and S the required area: then

$$\text{S} = ph \text{ (Prisms, \&c., Prop. 11.)}$$

If the area of the whole surface of a prism is required, the areas of the base and top of the prism, or (since these areas are equal: Prisms, &c., def. 2.) twice the area of the base of the prism, must be added to ph, the area of the lateral surface.

The area of the base is found by one of the rules for the determination of the areas of plane rectilineal figures in Section VIII.

If the base of the prism is a regular polygon, and r', the perpendicular from the centre on a side of this polygon (or the radius of the inscribed circle), the area of one end $=\frac{pr'}{2}$ (Art. 83. a.); the area of both ends $=pr'$; and the area of the whole surface of the prism $=ph+pr'=p(h+r')$.

If the prism is a cube whose edges are each equal to c;

$p=4c$; $h=c$; $r'=\frac{c}{2}$: and the expression $p(h+r')$ becomes

$$4c\left(c+\frac{c}{2}\right)=4c^2+2c^2=6c^2.$$

Whence the area of the surface of a cube is equal to six times the square of one of its edges; or to six times the area of its base.

Examples. 1st. Given the sides of the base of an upright triangular prism, equal to 7, 8, and 9 feet respectively, and the length of a lateral edge equal to 18 feet: required the area of the surface of the prism.

CALCULATION.

$p=7+8+9=24$ feet; $h=18$ feet; $\frac{p}{2}=s=12$ feet.

$\therefore ph=24\times 18=432$ sq. ft. $=$ area of the lateral surface.

Also, area of base $=\sqrt{\{12(12-7)(12-8)(12-9)\}}=\sqrt{(12\times5\times4\times3)}=\sqrt{720}=26.8328$,

Wherefore the area of both ends $=26.8328\times2=53.6656$ sq. ft.

And the area of the whole surface of the prism $=432+53.6656$ sq. ft. $=485.6656$ sq. ft.

2. Given the length of an upright parallelepiped = 25 feet; two adjacent sides of the base equal to 12.6 and 8.4 feet, respectively, and the angle included by these sides = 56° 18′: required the area of the surface of the parallelepiped.

Ans. 1226.1082 sq. ft.

3. In an upright parallelepiped with a square base, given a side of the base $=3\frac{1}{2}$ ft.; and one of the lateral edges = 12 ft.: required the area of the surface of the parallelepiped.

Ans. $192\frac{1}{2}$ sq. ft.

4. Given one of the edges of a cube $=10\frac{1}{2}$ inches: required the area of the surface of the cube.

Ans. 4 sq. ft. $85\frac{1}{2}$ sq. in.

5. If the length of an upright prism, which has a regular pentagon for its base, is 35 ft., and a side of the base is 7.3 ft.: what is the area of the surface of the prism?

Ans. 1460.8687 sq. ft.

a. If the prism is oblique, let a point be taken in one of the lateral edges, and let a plane at right angles to this edge be made to pass through the point. This plane is at right angles to all the lateral edges, (Planes, Prop. 7.). The section of the prism is a plane figure, having the same number of sides as the base; and its successive sides measure the distances between the two corresponding adjacent edges of the prism. The area of any face of the prism is equal to the product of the length of the side of the section, contained in that face by the length of one of the lateral edges of the prism, (Art. 75.) Whence, since the lateral edges are all of the same length, and the sum of the perpendicular distances between the whole of these edges composes the perimeter of the figure formed by the plane which cuts the prism at right angles to the lateral edges; it follows that the area of the lateral surface of an oblique prism is equal to the product of the perimeter of a section perpendicular to the lateral edges, by the length of one of these edges.

91. To find the area of the surface of an upright pyramid which has a regular polygon for its base.

Let the perimeter of the base of a pyramid $(=p)$, the straight line drawn from the vertex in the plane of one of the lateral faces, perpendicular to a side of the base $(=h')$, and the perpendicular drawn from the centre of the base to a side of the base $(=r')$ be given to find the area of the surface of the pyramid.

The area of the lateral surface of the pyramid $=\frac{ph'}{2}$ (Prisms, &c., Prop. 16.)

The area of the base . . . . . . $=\frac{pr'}{2}$ (Art. 83. *a.*)

Whence S, the area of the whole surface of the pyramid, is expressed by the formula

$$\text{S}=\frac{p\,(h'+r')}{2}.$$

Example. Given $p=6$ ft. 2 in.; $h'=15.02$ ft.; $r'=9.25$ in.: required S.

CALCULATION.

$$p=74\text{ in.};\ h'=180.24\text{ in.};\ r'=9.25\text{ in.}$$

$$\therefore \text{S}=\frac{74(180.24+9.25)}{2}=7011.13\text{ sq. in.}=48.6884+\text{sq. ft.}$$

a. If n, the number of sides of the base; l, the length of a side of the base; and h, the length of the axis (or the height) of the pyramid, are given to find S.

1st. $p=nl.$

2d. $r'=\frac{l}{2}\cot.\frac{180^\circ}{n}$, (Art. 83. *b.*)

3d. $h'=\sqrt{\{h^2+r'^2\}}$ (Euc. I. 47.) $=\sqrt{\left\{h^2+\left(\frac{l}{2}\cot.\frac{180^\circ}{n}\right)^2\right\}}$,

$$\text{Whence } \frac{ph'}{2}=\frac{nl}{2}\sqrt{\left\{h^2+\left(\frac{l}{2}\cot.\frac{180^\circ}{n}\right)^2\right\}},$$

and S, or $\frac{p(h'+r')}{2}=\frac{nl}{2}\left[\sqrt{\left\{h^2+\left(\frac{l}{2}\cot.\frac{180^\circ}{n}\right)^2\right\}}+\frac{l}{2}\cot.\frac{180^\circ}{n}\right]$.

Example.—Given $n=6$; $l=3$ feet; $h=9$ feet: required S.
Ans. $\text{S}=107.6904$ sq. ft.

CALCULATION.

1st. $p=nl=6\times3=18$ feet $=$ perimeter of the base of the pyramid.

2d. $r'=\frac{l}{2}\cot.\frac{180^\circ}{n}=\frac{3}{2}\cot.\frac{180^\circ}{6}=1.5\times\cot.30^\circ.$

$$\therefore\ \log.r'=\left\{\begin{array}{lr}\log.1.5\ \ldots\ldots & 0.176091\\ +\log.\cot.30^\circ\ \ldots & +\ 10.238561\\ -10\ \ldots\ldots & -\ \underline{10.000000}\end{array}\right.$$

The number of this log. is 2.5981 . . . 0.414652

$$\therefore\ r'=2.5981.$$

3d. $h'=\sqrt{\{h^2+r'^2\}}=\sqrt{\{9^2+2.5981^2\}}=\sqrt{87.75}=9.3675.$

4th. $\text{S}=\frac{p\,(h'+r')}{2}=18\,(9.3675+2.5981)=107.6904$ sq. ft.

b. If n, the number of sides of the base; l, the length of a side of the base; and e, the length of a lateral edge of the pyramid, are given to find S, 1st. $p = nl$.

2d. $r' = \frac{l}{2} \cot. \frac{180°}{n}$.

3d. $h' = \sqrt{\left\{e^2 - \frac{l^2}{4}\right\}}$,

$$\therefore \frac{ph'}{2} = \frac{nl}{2}\sqrt{\left\{e^2 - \frac{l^2}{4}\right\}},$$

$$\text{And s or } \frac{p(h'+r')}{2} = \frac{nl}{2}\left[\sqrt{\left\{e^2 - \frac{l^2}{4}\right\}} + \frac{l}{2}\cot. \frac{180°}{n}\right].$$

Example.—Given $n = 8$; $l = 6$ ft.; $e = 16$ ft.: required s.
Ans. s $= 551.01264$ sq. ft.

CALCULATION.

1st. $p = nl = 8 \times 6 = 48$ ft.

2d. $r' = \frac{l}{2} \cot. \frac{180°}{n} = \frac{6}{2} \cot. \frac{180°}{8} = 3 \cot. 22° \, 30' = 7.24263$.

3d. $h' = \sqrt{\left\{e^2 - \frac{l^2}{4}\right\}} = \sqrt{\left\{16^2 - \frac{6^2}{4}\right\}} = \sqrt{247} = 15.71623$.

4th. s $= \frac{p(h'+r')}{2} = \frac{48}{2}(15.71623 + 7.24263) = 551.01264$ sq. ft.

Other examples. 1st. Given $n = 4$; $l = 3$ ft.; $h = 24$ ft.: required s.
Ans. s $= 153.2808$ sq. ft.

2d. Given $n = 6$; $l = 10$ ft.; $h = 45$ ft.: required s.
Ans. s $= 1634.5803$ sq. ft.

3d. Given $n = 12$; $l = 4$ ft.; $e = 50$ ft.: required s.
Ans. s $= 1378.176$ sq. ft.

92. To find the area of the surface of an upright cylinder.

Given the radius of the base and the length of the axis (or the height) of an upright cylinder to find the area of the surface of the cylinder.

Let r be the radius of the base, and h the height of a cylinder: and let the area of its surface be denoted by s.

Then the area of the convex surface of the cylinder is equal to $h \times$ circumference r, (Cylinder, &c. Prop. 7.),

or, since circumference $r = 2\,\pi r$, (Art. 84.),
the area of the convex surface of the cylinder $= 2\,\pi r h$.

Again, the ends of the cylinder being equal circles whose common radius is r, $\pi r^2 =$ the area of one end (Art. 86.), and $2\,\pi r^2 =$ the area of both ends;

Whence the area of the whole surface of the cylinder $= 2\pi r h + 2\pi r^2$,

$$\text{or } S = 2\pi r(h + r).$$

Consequently, to find the area of the whole surface of a cylinder:—Multiply the sum of the values of the height and radius by the circumference of the cylinder: the product expresses the area of the whole surface of the cylinder.

Examples.—1st. Given $r = 1.5$ ft.; $h = 40$ ft.: required S.
Ans. $S = 492.401$ sq. ft.

CALCULATION.

$$h + r = 40 + 1.5 = 41.5,$$

$$\therefore S = 2\pi r(h + r) = 2\pi \times 1.5 \times 41.5 = \pi \times 3 \times 41.5,$$

$$\text{And } \log. S = \begin{cases} \log. \pi \quad . \quad . \quad . \quad . \quad . \quad . & 0.497150 \\ + \log. 3 \quad . \quad . \quad . \quad . \quad . \quad . + & 0.477121 \\ + \log. 41.5 \quad . \quad . \quad . \quad . \quad . + & 1.618048 \end{cases}$$

The number of this log. is 492.401 . . . 2.692319

$$\therefore S = 492.401 \text{ sq. ft.}$$

2d. What quantity of sheet iron is required to make a cylindrical funnel 22 inches in diameter and 36 feet in length.
Ans. 207.3456 sq. ft.

3d. Required the area of the concave surface and bottom of a cylindrical reservoir which is 18 feet in diameter and 12 feet in depth.
Ans. 933.0552 sq. ft.

4th. Required the area of the whole surface of a cylindrical pillar which is 5½ feet in diameter and 45 feet in height.
Ans. 825.0627 sq. ft.

a. If the height of a cylinder is equal to the diameter of its base; that is, if $h = 2r$,

the area of the convex surface $= 2\pi r \times 2r$,
and the area of the two ends $= 2\pi r \times r$,

$$\therefore S = 2\pi r(2r + r) = 2\pi r \times 3r = 6\pi r^2.$$

A cylinder whose height is equal to the diameter of its base is termed equilateral.

If the area of the base of an equilateral cylinder is πr^2, the area of the convex surface is $4\pi r^2$, and the area of the whole surface $6\pi r^2$. Whence, in an equilateral cylinder, the area of the convex surface is equal to four times, and the area of the whole surface is equal to six times the area of the circle which forms the base of the cylinder.

93. To find the area of the surface of an upright cone.

Given the radius of the base and the height (or length of the axis) of an upright cone to find the area of the surface of the cone.

Let r be the radius of the base, and h the height of the cone: also let the side (or slant height) of the cone be denoted by h'; and the required area by s.

Then since h' is the hypotenuse of a right-angled triangle of which r is the base and h the perpendicular, $h' = \sqrt{h^2 + r^2}$. (Euc. I. 47.)

Also $\pi r =$ half the circumference of the base of the cone, (Art. 84.);

$\therefore$ $\pi r h'$ or $\pi r \sqrt{h^2 + r^2} =$ the area of the convex surface of the cone, (Cylinder, &c. Prop. 9.).

Again, πr^2 or $\pi r \times r =$ the area of the base of the cone, (Art. 86.),

$$\therefore \ \mathrm{s} = \pi r \times h' + \pi r \times r = \pi r\,(h' + r):$$

in words:—The area of the convex surface of an upright cone is equal to the area of a rectangle of which one side is equal in length to half the circumference of the base of the cone, and the other to the side (or slant height) of the cone. And the area of the whole surface of an upright cone is equal to the area of a rectangle of which one side is equal in length to half the circumference of the base of the cone, and the other to a straight line made up of the side and the radius of the base of the cone.

Examples. 1st. Given $r = 2\frac{1}{2}$ ft.; $h = 9$ ft.: required s.
Ans. s $= 92.9974$ sq. ft.

CALCULATION.

$$h' = \sqrt{\{2.5^2 + 9^2\}} = \sqrt{87.25} = 9.34077,$$

$$h' + r = 9.34077 + 2.5 = 11.84077,$$

$$\mathrm{s} = \pi r\,(h' + r) = 3.1416 \times 2.5 \times 11.84077 = 92.99740758 \text{ sq. ft.}$$

2d. How many square yards of canvas are required to make a conical tent, which is 20 feet in diameter and 12 feet high?

Ans. 54.5259587 square yards.

3d. The circumference of the base of a conical spire is 64 feet, and the perpendicular height 118 feet: required the area of the convex surface of the spire.

Ans. 3790.04192 square feet.

4th. The lengths of the diameter of the base and of the side of a conical buoy are each 12 feet: required the area of the whole surface of the buoy.

Ans. 339.2928 square feet.

5th. The radius of the base of an upright cone is r, and the altitude $2\,r$: required the expressions of the areas of the convex surface and the whole surface of the cone.

Ans. Area of convex surface $= \pi r^2 \sqrt{5}$. Area of whole surface $= \pi r^2 (1 + \sqrt{5})$.

a. If $h' = 2\,r$, $\pi r h' = \pi r \times 2\,r = 2\,\pi r^2$;

$$\text{and } \pi r\,(h' + r) = \pi r \times 3\,r = 3\,\pi r^2.$$

A cone whose side is equal to the diameter of its base is termed equilateral.

Hence, the area of the convex surface of an equilateral cone is equal to twice, and the area of the whole surface is equal to three times, the area of the circle which forms the base of the cone.

94. To find the area of the surface of a frustum of an upright cone.

Given the radius of the base, the radius of the top, and the length of the side of a frustum of an upright cone, to find the area of the surface of the frustum.

Let r be the radius of the base; r' the radius of the top; h' the length of the side; and s the required area;

Then $2\,\pi r =$ the circumference of the base of the frustum (Art. 84.),

and $2\,\pi r' =$ the circumference of the top,

$\therefore\ \pi\,(r + r') =$ half the sum of the circumferences of the base and top:

whence $\pi r h' + \pi r' h'$ or $\pi\,(r + r')\,h' =$ the area of the convex surface of the frustum, (Cylinder, &c. Prop. 10.).

Again, $\pi r^2 =$ the area of the base of the frustum, (Art. 86.) and $\pi r'^2 =$ the area of the top,

$$\therefore s = \pi (r + r')h' + \pi r^2 + \pi r'^2 = \pi\,\{r\,(h' + r) + r'(h' + r')\},$$

$$\text{or } s = \pi r\,(h' + r) + \pi r'\,(h' + r'):$$

Whence the convex surface of a frustum of a cone is equal to the convex surface of two cones of which the one has the base, and the other the top of the frustum, for their respective bases; the side of each being equal to the side of the frustum; and the whole surface of the frustum is equal to the whole surface of the same cones.

Examples. 1st. Given $r = 9$ in.; $r' = 4\frac{1}{2}$ in.; $h' = 14$ ft. 3 in.: required s.
Ans. s $= 52.57271$ sq. ft.

CALCULATION.

$$h' = 14 \text{ ft. } 3 \text{ in.} = 171 \text{ in.}$$

$$\therefore\ h' + r = 171 + 9 = 180; \text{ and } r\,(h' + r) = 9 \times 180 = 1620,$$

$$h' + r' = 171 + 4.5 = 175.5; \text{ and } r'\,(h' + r') = 4.5 \times 175.5 = 789.75,$$

$$\therefore\ r\,(h' + r) + r'\,(h' + r') = 1620 + 789.75 = 2409.75,$$

And s or $\pi\left\{r(h' + r) + r'(h' + r')\right\} = 3.1416 \times 2409.75 = 7570.4706$ sq. in. $= 52.57271$ sq. ft.

2d. Given $r = 12$ ft.; $r' = 6$ ft.; $h' = 10$ ft.: required the area of the convex surface of the frustum.

Ans. 565.488 sq. ft.

3d. The diameter of the base of a frustum of a cone is 5 ft.; the diameter of the top 3 ft.; and the length of the side 45 ft.: required the area of the whole surface of the frustum.

Ans. 592.1916 sq. ft.

a. If r, r', the radii of the base and top, and h the altitude (or length of the axis) of the frustum, are given; h' is determined by the formula $h' = \sqrt{\{h^2 + (r - r')^2\}}$.

For example, let $r = 12$; $r' = 6$; $h = 8$: then

$$h' = \sqrt{\{8^2 + (12 - 6)^2\}} = \sqrt{64 + 36} = 10.$$

b. If $\pi\,(r + r')$ express the circumference of the base of a cylinder, and h' its height, $\pi\,(r + r')\,h'$ expresses the area of the convex surface of the cylinder. Whence the area of the convex surface of a conical frustum is equal to the area of the convex surface of a cylinder whose height is equal to the side of the frustum, and the circumference of its base to half the sum of the circumferences of the base and top of the frustum.

If $r = r'$, the frustum becomes a cylinder, and

$$\pi\,(r + r)\,h = 2\,\pi r h.$$

95. To find the area of the surface of a sphere.

Given the radius of a sphere to find the area of the surface of the sphere.

Let r be the radius, and s the required area.

It is proved, (Cylinder, &c., Prop. 14.) that the surface of a sphere is equivalent to the convex surface of the equilateral cylinder described about the sphere.

Again, it is proved, (Art. 92. *a.*) that the area of the convex surface of an equilateral cylinder, the radius of whose base is r, is given by the expression $4\pi r^2$:

whence, if the radius of a sphere $= r$; s $= 4\pi r^2$.

a. Since πr^2 expresses the area of a great circle of the sphere whose radius is r, it follows that the area of the surface of any sphere is equal to four times the area of a great circle of that sphere.

b. The expression $4\pi r^2$ may be separated into the two factors $2\pi r$ and $2r$, of which the first gives the circumference, and the second the diameter, of the circle or the sphere whose radius is r: hence another rule for finding the area of the surface of a sphere, viz. Multiply the circumference of the sphere by its diameter, the product is the area of the sphere.

c. The area of the whole surface of the equilateral cylinder described about the sphere whose radius is r, is $6\pi r^2$, (Art. 92. *a.*). Whence the area of the surface of a sphere, is to the area of the cylinder described about it, as 4 to 6, or 2 to 3.

Examples. 1st. The diameter of a sphere is 12 ft.: required s.
Ans. s $=452.3904$ sq. ft.

CALCULATION.

$$r=6 \text{ and } r^2=36,$$

$$\therefore \text{s or } 4\pi r^2=4\times 3.1416\times 36=3.1416\times 144=452.3904 \text{ sq. ft.}$$

2d. Given $d=1$ ft. 8 in.: required s.
Ans. s $=8.7266$ sq. ft.

3d. Given $r=15$ ft.: required s.
Ans. s $=2827.44$ sq. ft.

d. If c, the circumference of a sphere, is given to find the area of the surface of the sphere; let d = the diameter of the sphere: then since $d=\frac{c}{\pi}$, $c\times d=c\times\frac{c}{\pi}=\frac{c^2}{\pi}$.

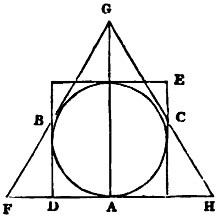
Consequently, if the length of the circumference of a sphere is given to find the area of the surface of the sphere; — divide the square of the circumference by π, the result expresses the area of the surface of the sphere.

Examples. 1st. Given $c=1$: required s.
Ans. s $=.3183099$.

2d. Given $c=50.3$: required s.
Ans. s $=805.3527$.

3d. In a cylindrical pontoon, with hemispherical ends, given the length of the cylindrical part $=l$ feet, the diameter $=d$ feet, and consequently the diameter of the ends also $=d$ feet: required the convex surface of the pontoon.
Ans. $\pi d\,(l+d)$ square feet.

4th. About a circle ABC, let a square DE and an equilateral triangle FGH be described, and let the circle, square, and triangle be turned round the diameter AG, producing respectively a sphere, an equilateral cylinder, and an equilateral cone: it is required to find the areas of the whole surfaces of the three solids; the radius of the circle ABC being r.



Ans. The area of the surface of the sphere $=4\pi r^2$,
. . cylinder $=6\pi r^2$,
. . cone $=9\pi r^2$.

96. To find the area of the convex surface of a segment or zone of a sphere.

Given the radius of a sphere, and the height of a segment or zone of that sphere, to find the area of the convex surface of the segment or zone.

Let r be the radius of the sphere, and h the height of the segment or zone.

Then, $2\pi r=$ the circumference of a great circle of the sphere, or of the base of the equilateral cylinder described about the sphere: and, by hypotheses, $h=$ the height of the segment or zone; whence the area of the convex surface of the segment or zone $=2\pi rh$. (Cylinder, &c., Prop. 14., Cor. 1.)

Examples. 1st. A segment 18 inches in height is cut from a sphere whose radius is 5 feet:- required the area of the convex surface of the segment.
Ans. 47.124 sq. ft.

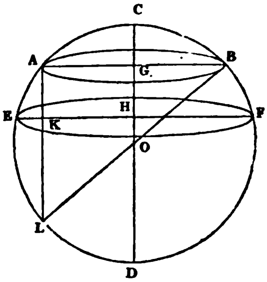
CALCULATION.

$2\pi r=2\times 3.1416\times 5=31.416=$ circumference of the sphere,
$h=18$ in. $=1\frac{1}{2}$ ft. $=1.5$ ft. $=$ height of the segment,

$\therefore 2\pi rh=31.416\times 1.5=47.124$ sq. ft. $=\left\{\begin{array}{l}\text{area of the convex surface}\\ \text{of the segment.}\end{array}\right.$

2d. The height of a segment of a sphere is $7\frac{1}{2}$ feet, and the radius of the sphere 10 feet: required the area of the convex surface of the segment.
Ans. 471.24 sq. ft.

3d. A zone 3 feet in height is cut from a sphere whose diameter is 16 ft.: required the area of the convex surface of the zone.
Ans. 150.7968 sq. ft.



a. Let ECF be a segment of a sphere, and let $EH=b$, the radius of the base; and $CH=h$, the height of the segment, be given to find the area of the convex surface of the segment.

Since the expression of the required area is $2\pi r h$, it becomes necessary to determine r in terms of b and h, the data of the problem.

Now, if CH, the axis of the segment, is produced, it is a diameter of the sphere or of the great circle ACD formed by a plane passing through CH and HE.

Whence $EH^2=CH\times HD$, (Euc. III. 35.)$=CH\,(CD-CH)$,

$$\text{or, } b^2=h\,(2r-h) \quad \therefore\ r=\frac{b^2+h^2}{2h},$$

$$\text{and } 2\pi r h=2\pi\times\frac{b^2+h^2}{2h}\times h=\pi\,(b^2+h^2)^*$$

Ex. Given the diameter of the base of a spherical segment = 28 ft. 9 in., and the height = 1 ft. 3 in.: required the area of the convex surface of the segment.

Ans. 654.0909375 sq. ft.

CALCULATION.

$$b=\tfrac{1}{2}\text{ of } 28\text{ ft. } 9\text{ in.}=14\text{ ft. } 4\tfrac{1}{2}\text{ in.}=14.375\text{ ft.}$$

$$h=1\text{ ft. } 3\text{ in.}=1\tfrac{1}{4}\text{ ft.}=1.25\text{ ft.}$$

$$b^2=14.375^2=206.640625:\ h^2=1.5625;$$

$$\therefore\ b^2+h^2=206.640625+1.5625=208.203125,$$

$$\text{and } \pi\,(b^2+h^2)=3.1416\times 208.203125=654.0909375\text{ sq. ft.}$$

b. If it is required to find the area of the whole surface of a spherical segment, the area of the circle which forms the base of the segment must be added to the area of the convex surface. The sum of these areas is the result required.

If r, the radius of the sphere, h, the height, and b, the radius of the base of the segment, are given to find the area of the whole surface of the segment;

$2\pi r h=$the area of the convex surface of the segment,
$\pi b^2=$ the area of the base,

$\therefore\ \pi\,(2rh+b^2)=$the area of the whole surface.

If r, the radius of the sphere, and h, the height of the segment, are the only data;

---

* $b^2+h^2=EC^2=\text{chord}^2$ of half the circumference of the segment ECF.

since $\text{EH}^2 = \text{CH}(\text{CD} - \text{CH})$ or,

$b^2 = h(2r - h)$, and therefore $b = \sqrt{\{h(2r-h)\}}$; the value of b may be determined from the values of r and h, and the area of the whole segment found as before.

If b and h, only, are given; $r = \dfrac{b^2+h^2}{2h}$, (Art. 96. *a.*)

c. Let EABF be a spherical zone, and let $\text{EH} = b$, the radius of the base of the zone; $\text{AG} = t$, the radius of the top of the zone; and $\text{HG} = h$, the height of the zone, be given to find the area of its convex surface.

The expression of the area of the convex surface of the zone being $2\pi r h$, it becomes necessary to express r in terms of the given quantities b, t, h.

Through A, let AKL be drawn perpendicular to EF,

Then, $\text{AK} = \text{HG} = h$; $\text{KH} = \text{AG}$; $\text{EK} = \text{EH} - \text{AG} = b - t$: $\text{KF} = \text{FH} + \text{KH} = \text{EH} + \text{AG} = b + t$.

Also, $\text{EK} \times \text{KF} = \text{AK} \times \text{KL}$ (Euc. III. 35.),

or, $(b-t)(b+t) = h \times \text{KL}$, $\therefore\ \text{KL} = \dfrac{b^2-t^2}{h}$,

$$\text{and AL } (= \text{AK} + \text{KL}) = h + \frac{b^2-t^2}{h} = \frac{h^2+b^2-t^2}{h}.$$

But since LAB is a right angle, LB, if joined, is a diameter, (Euc. III. 31.),

$$\therefore\ \text{LB}^2 = \text{AB}^2 + \text{AL}^2, \text{ (Euc. I. 47.)},$$

$$\text{or } d^2 = 4t^2 + \left(\frac{h^2+b^2-t^2}{h}\right)^2$$

$$\therefore\ d = \sqrt{\left\{4t^2 + \left(\frac{h^2+b^2-t^2}{h}\right)^2\right\}},$$

$$r = \tfrac{1}{2}\sqrt{\left\{4t^2 + \left(\frac{h^2+b^2-t^2}{h}\right)^2\right\}},$$

$$\text{And } 2\pi r h = 2\pi h \times \tfrac{1}{2}\sqrt{\left\{4t^2 + \left(\frac{h^2+b^2-t^2}{h}\right)^2\right\}} = \pi\sqrt{\{4h^2t^2 + (h^2+b^2-t^2)^2\}},$$

which is the expression required.

d. If the area of the whole surface of a spherical zone is required, the areas of the base and top of the zone must be added to the area of its convex surface.

Let r be the radius of the sphere; b, the radius of the base of the zone; t, the radius of its top; and h, its height.

Then $2\pi r h =$ the area of the convex surface of the zone,

$\pi b^2 =$ the area of the base,

$\pi t^2 =$ the area of the top,

$\therefore\ \pi(2rh + b^2 + t^2) =$ the area of the whole surface of the zone.

97. To find the volume of a parallelepiped.

In a parallelepiped, let there be given the length of a side of the base, the perpendicular distance between the given and the opposite side (or the breadth) of the base, and the height, to find the volume of the parallelepiped.

$$\text{Let } l \text{ be the length of the base} \left.\begin{array}{l} \\ b \text{ the breadth of the base} \\ h \text{ the height} \end{array}\right\} \text{of the parallelepiped,}$$

and let the volume be denoted by V.

Then $V = lbh$ (Art. 7. *g.*).

If $l = b$; $V = l^2h$: and if $l = b = h$; $V = l^3$.

If the base is rectangular, and the parallelepiped upright, *l, b, h,* are three contiguous edges of the solid.

Examples. 1st. Given $l = 3$ ft. 8 in.; $b = 2$ ft. 4 in.; $h = 16$ ft. 10 in.: required V.

Ans. V = 144 cub. ft. 32 cub. in.

CALCULATION (App. Art. 14.).

$$\begin{array}{rcrcrcrl}
3 & . & 8 & & & & & = \text{length of base.}\\
2 & & 4 & & & & & = \text{breadth of base.}\\ \hline
7 & . & 4 & & & & & \\
1 & . & 2 & . & 8 & & & \\ \hline
8 & . & 6 & . & 8 & & & = \text{area of base.}\\
16 & . & 10 & & & & & = \text{height.}\\ \hline
136 & . & 10 & . & 8 & & & \\
7 & . & 1 & . & 6 & . & 8 & \\ \hline
144 & . & 0 & . & 2 & . & 8 &
\end{array}$$

$$\text{or, } 144 + \frac{2}{144} + \frac{8}{1728} \text{ cubic feet,}$$

or, 144 cubic feet, 32 cubic in. = volume of the proposed parallelepiped.

2d. Given $l = 2$ ft. 9 in.; $b = 9$ in.; $h = 25$ ft.: required V.

Ans. V = 51 cubic ft. 972 cubic in.

3d. If the height of a square pillar is 18 ft., and the sides of its base 1.25 ft., what is the solid content of the pillar?

Ans. 28.125 cubic ft.

4th. If one of the edges of a cube measures 10 ft. 4 in., what is the volume of the cube?

Ans. 1103$\frac{10}{27}$ cubic ft.

5th. The length and breadth of the base of an oblique parallelepiped are 7½ and 5 ft. respectively, and the lateral edges, which are 4 ft. in length, are inclined at an angle of 42° to the plane of the base: required the volume of the parallelepiped.

Ans. 100.36875 cubic ft.

a. Let the lateral edges (and consequently the lateral faces) of an oblique parallelepiped be produced indefinitely;

let a point be taken in the produced part of one of the edges, and through this point let a plane be made to pass perpendicular to the lateral edges: this plane is perpendicular to the lateral faces of the parallelepiped (Planes, Prop. 18.), and the area of the part of it inclosed by these faces is equal to the area of a section of the parallelepiped made at right angles to its lateral edges.

In the produced part of one of the lateral edges, and from the point in which that edge is cut by the perpendicular plane, let a distance be taken equal to the length of a lateral edge of the oblique parallelepiped, and through the extremity of this distance let another plane be made to pass perpendicular to the parallel edges. The produced lateral faces and the two planes at right angles to them form an upright parallelepiped whose base is equal to a section of the oblique parallelepiped, made by a plane at right angles to the lateral edges, and its lateral edges of the same length as those of the oblique parallelepiped.

The two parallelepipeds are equivalent (Prisms, &c. Prop. 2.), and the volume of the upright parallelepiped is equal to the product of the area of its base by its altitude (Art. 97.): whence the volume of an oblique parallelepiped is equal to the product of the area of a section which is perpendicular to the lateral edges, by the length of one of the lateral edges.

b. If the oblique and upright parallelepiped are both cut by the same diagonal plane, each is cut into two triangular prisms of equal volume (Prisms, &c. Prop. 8.). Consequently, an oblique triangular prism is equivalent to an upright triangular prism whose base is equal to a section of the oblique prism, made at right angles to the lateral edges, and its height equal to the length of a lateral edge of the oblique prism.

c. By producing the lateral edges of any oblique prism, with a polygonal base, constructing an upright prism having the produced lateral faces of the oblique prism for its lateral faces, and lateral edges of the same length as the oblique prism; and imagining diagonal planes to divide both the oblique and the upright prism into triangular prisms, it can be in like manner proved that any oblique prism is equivalent to an upright prism whose base is equal to a section of the oblique prism made at right angles to the lateral edges, and its height equal to the length of a lateral edge of the oblique prism.

98. To find the volume of a prism.

Since a prism is equivalent to a parallelepiped which has an equivalent base and an equal altitude (Prisms, &c.

Prop. 9.); and the volume of a parallelepiped is obtained by multiplying the area of its base by its altitude (Art. 7. *g.*), it follows that the volume of a prism is obtained by,

1st. Finding the area of the base of the prism by the proper rule (Section VIII.); and, 2d, multiplying this area by the altitude of the prism.

If b^2 express the area of the base of a prism, and h the height, the volume of the prism is $b^2 h$.

Examples. 1st. Given the three sides of the base of a triangular prism equal to 5, 7, and 8 feet, respectively, and the height of the prism equal to 16 feet: required the volume of the prism.

Ans. 277.128 cubic feet.

CALCULATION.

$5+7+8=20$: and $20 \div 2=10=$ semiperimeter of the base of the prism,

$\left.\begin{array}{l}10-5=5\\10-7=3\\10-8=2\end{array}\right\}=$ excesses of the semiperimeter over the sides of the base,

$\sqrt{\{10 \times 5 \times 3 \times 2\}}=\sqrt{300}=17.3205=$ area of the base of the prism,

$17.3205 \times 16=277.128$ cubic feet $=$ volume of the proposed prism.

2d. The base of a triangular prism is an equilateral triangle whose sides are each equal to 1 ft. 5 in.; and the height of the prism is 12 ft. 5 in.: required its volume.

Ans. 10.7905 cubic feet.

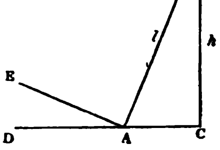
3d. How many gallons of water will fill a reservoir which is 30 ft. deep, 60 ft. wide at top, 48 ft. wide at bottom, and 150 ft. long.

Ans. 1514400.917 gallons.

4th. The circumference of a hexagonal pillar is 7 ft., and the height 11 ft. 2 in.: required the solid content of the pillar.

Ans. 39.48835 cubic feet.

a. It follows from Art. 97. *c.*, and the preceding part of this article, that the volume of an oblique prism is equal to the product of the area of a section of the prism, made at right angles to the lateral edges, by the length of one of the lateral edges.



b. Let AB be one of the lateral edges of an oblique prism: from B let BC be drawn perpendicular to the plane of the base (Planes, Prop. 10.), and let BC pierce that plane in C: then BC is the altitude of the prism.

Through AB, BC let a plane, BCD, pass, cutting the plane of the base in AD, and the plane of the section which is at right angles to the lateral edges of the prism in AE.

The plane BCD is at right angles to the plane of the base (Planes, Prop. 18.): the angle BAC measures the inclination of the prism to the plane of its base; and the angle DAE measures the inclination of the plane of the base to the plane of the section which is at right angles to the lateral edges of the prism. Also BAE is a right angle: therefore BAC and DAE are together equal to a right angle (Euc. I. 13.): therefore the angles BAC, DAE are complements of each other.

Let the area of the base of the prism be denoted by b^2;
the area of the section which is at right angles to the lateral edges of the prism, by s^2;
the inclination of the plane of the base to the plane of the section by i;
the length of a lateral edge (AB) of the prism by l;
and the altitude (BC) by h;

Then by Art. 97. *c.*, $b^2 \times h = s^2 \times l$.

But, $h = l \sin . \mathrm{BAC} = l \cos . \mathrm{DAE} = l \cos i$, (Art. 56.)

$$\therefore\ b^2 \times l \cos . i = s^2 \times l, \text{ or } b^2 \cos . i = s^2.$$

Hence the area of a section of an oblique prism, made by a plane perpendicular to the lateral edges of the prism, is obtained by multiplying the area of the base of the prism by the cosine of the inclination of the plane of the base to the plane of the section.

Hence, also, if from all the angles of a given plane polygon perpendiculars are let fall upon another plane, and the feet of the perpendiculars are joined so as to form a second polygon of the same number of sides as the given polygon; the area of the second polygon is obtained by multiplying the area of the given polygon by the cosine of the inclination of the plane in which it is situated to the plane of the second polygon.*

99. To find the volume of a pyramid.

Given the area of the base and the height of a pyramid to find the volume of the pyramid.

Let b^2 be the area of the base of a pyramid; h the height; and V the required volume.

Then $b^2 \times h =$ the volume of the prism whose base is b^2, and height h, (Art. 98.),

$\therefore\ \frac{b^2 \times h}{3} =$ the volume of the pyramid whose base is b^2, and height h, (Prisms, &c. Prop. 15. cor. 1.): in words: — Multiply the area of the base of a pyramid by the height, and divide the product by 3; the result expresses the volume of the pyramid.

---

* The second polygon is termed the projection of the given polygon.

Examples. 1st. The base of a pyramid is an equilateral triangle whose side measures 6 feet; and the height of the pyramid is 16 feet: required the volume of the pyramid.
Ans. $V = 83.1384$ cubic feet.

CALCULATION.

$$\text{Area of base} = \frac{6^2}{4}\sqrt{3}\ (\text{Art. } 76.\, e.) = 9\sqrt{3} = 15.58845,$$

$$\therefore\ V = \frac{15.58845 \times 16}{3} = 83.1384 \text{ cubic feet.}$$

2d. The sides of the base of a triangular pyramid are 13 ft., 14 ft., and 15 ft., and its height is 63 ft.: required the volume of the pyramid.
Ans. $V = 1764$ cubic feet.

3d. The base of a pyramid is a regular octagon whose side measures 4 ft. 11 in., and the height of the pyramid is 41 ft.: required the volume of the pyramid.
Ans. $V = 1595.1804$ cubic feet.

100. To find the volume of a cylinder.

Given the radius of the base and the height of a cylinder to find its volume.

Let r be the radius of the base, and h the height of a cylinder:

Then $\pi r^2 =$ the area of the base of the cylinder, (Art. 86.) and $\pi r^2 h =$ the volume of the cylinder. (Cylinder, &c. Prop. 5.)

If the calculation is made with logarithms,

$$\log. V = \log. \pi + 2 \log. r + \log. h.$$

Examples. 1st. Given $r = 1$ ft. 6 in.; $h = 15$ ft.: required V.
Ans. $V = 106.02867$ cubic feet.

CALCULATION.

$$\text{Log. } V = \begin{cases} \log. \pi \quad . \quad . \quad . \quad . \quad 0.497150 \\ +2 \log. 1.5 \quad . \quad . \quad . \quad + \ 0.352182 \\ +\log. 15 \quad . \quad . \quad . \quad . \quad + \ 1.176091 \end{cases}$$

The number of this log. is 106.02867 . . 2.025423

$\therefore$ the volume of the proposed cylinder is 106.02867 cubic feet.

2d. Given $r = 10\frac{3}{4}$ in.; $h = 16$ ft.: required V.
Ans. $V = 40.339$ cubic feet.

3d. Given the circumference of a cylinder $= 8$ ft., and the height 18 ft.: required the volume of the cylinder.
Ans. $V = 91.676$ cubic feet.

101. To find the volume of a cone.

Given the radius of the base and the height of a cone to find the volume of the cone.

Let r be the radius of the base, and h the height of a cone.

Then $\pi r^2 =$ the area of the base of the cone, (Art. 86.) and $\pi r^2 h =$ the volume of the cylinder which has the base πr^2, and the height h, (Art. 100.)

$\therefore\ \frac{\pi r^2 h}{3} =$ volume of the cone. (Cylinder, &c. prop. 6. cor.)

Examples. 1st. Given $r = 1$ ft. $1\frac{1}{2}$ in.; $h = 16\frac{1}{2}$ ft.: required V.

Ans. V $= 21.8684$ cubic feet.

2d. Given the circumference of the base of a cone $= 9$ ft., and the height $= 10.5$ ft.: required the volume of the cone.

Ans. V $= 22.5609$ cubic feet.

3d Given the circumference of the base of a cone $= 24$ ft., and the length of a side of the cone $= 32$ ft.: required the volume of the cone.

Ans. V $= 485.428$ cubic feet.

102. To find the volume of a sphere.

Given the radius of a sphere to find its volume.

Let r be the radius of a sphere.

Then $\pi r^2 =$ the area of a great circle of the sphere or the area of the base of an equilateral cylinder described about the sphere.

Also $2r =$ the height of this cylinder,

$\therefore\ \pi r^2 \times 2r = 2\pi r^3 =$ the volume of the cylinder described above the sphere.

$\therefore\ \frac{2}{3}$ of $2\pi r^3 = \frac{4}{3}\pi r^3 =$ the volume of the sphere whose radius is r (Cylinder, &c., Prop. 12.).

The expression $\frac{4}{3}\pi r^3$ may be resolved into the two factors $4\pi r^2$ and $\frac{1}{3}r$; of which $4\pi r^2$ expresses the convex surface of the sphere (Art. 95.), and $\frac{1}{3}r$, the third part of its radius: whence the volume of a sphere is obtained by multiplying the surface of the sphere by the radius, and dividing the product by 3.

$4\pi r^2 \times \frac{1}{3}r$ also expresses the volume of a pyramid whose base is $4\pi r^2$ and height, r. Whence the volume of a sphere is equal to that of a pyramid, the area of whose base is equal to the area of the surface of the sphere, and its height equal to the radius of the sphere.

Examples. 1st. Given $r = 10$ inches: required V.

Ans. V $= 2.424$ cubic feet.

CALCULATION.

$$r = 10 \quad \therefore\ r^3 = 10^3 = 1000,$$

$$\therefore\ \text{V} = \frac{4}{3}\pi r^3 = \frac{4}{3} \times 3.1416 \times 1000 = 4188.8 \text{ cub. in.} = 2.424 \text{ cub. ft.}$$

2d. Required the volume of a sphere whose diameter is 30 feet.

Ans. 14137.2 cubic feet.

3d. Required the volume of a sphere whose circumference is 50.3 feet.

Ans. 2149.07373 cubic feet.

4th. Required the volume of the pontoon described in the 3d example of Art. 95.

Ans. $\frac{1}{4}\pi d^2 \left(l \times \frac{2d}{3}\right)$.

5th. Required the volumes of the sphere, cylinder, and cone described in the 4th example of Art. 95.

Ans. The volume of the sphere $= \frac{4}{3}\pi r^3$.
. . . cylinder $= 2\pi r^3$.
. . . cone $= 3\pi r^3$.

a. If the diameter of a sphere is 1, the radius is $\frac{1}{2}$ and the volume$= \frac{4}{3} \times \pi \times (\frac{1}{2})^3 = \frac{4}{3} \times \pi \times \frac{1}{8} = \frac{1}{6} \times 3.1416 = .5236$. Let d be the diameter of another sphere, whose volume may be denoted by x.

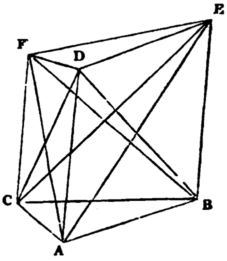
Then, since the volumes of spheres are proportional to the cubes of their diameters (Cylinder, &c., Prop. 12. Cor. 1.),

$$1^3 : d^3 :: .5236 : x,$$
$$\therefore x = d^3 \times .5236.$$

Whence if the cube of the diameter of a sphere is multiplied by .5236, the result expresses the volume of the sphere.

103. To find the volume of a frustum of a triangular prism.

Let ABCDEF be a frustum of a triangular prism. Through the three points D, C, B, let a plane be made to pass; and through the three points D, C, E, let another plane be made to pass. The frustum is divided, by these planes, into the three triangular pyramids D–ABC; D–BCE; C–EDF.



1st. D–ABC has ABC for base and D for vertex.

2nd. D–BCE may be transformed into another pyramid, A–BCE (Prisms, &c., Prop. 14.),

And A–BCE may be considered as having ABC for base and E for vertex.

3d. C–EDF, may be considered as having CEF for base and D for vertex; and

D–CEF may be transformed into another pyramid, A–CEF (Prisms, &c., Prop. 14.); also,
A–CEF may be considered as having ACF for base and E for vertex:
E–ACF may be transformed into the pyramid B–ACF (Prisms, &c., Prop. 14.); and
B–ACF may be considered as having ABC for base and F for vertex.

Hence the three pyramids which compose the frustum are equivalent to the three pyramids D–ABC; E–ABC; F–ABC.

Now, the volume of D–ABC = area ABC × $\frac{1}{3}$ of the perpendicular from D, on the plane ABC (Art. 99.);
the volume of E–ABC = area ABC × $\frac{1}{3}$ of the perpendicular from E on the plane ABC;
the volume of F–ABC = area ABC × $\frac{1}{3}$ of the perpendicular from F on the plane ABC.

Wherefore the volume of the frustum ABCDEF is obtained by multiplying the area of one of its bases (ABC), by one-third of the sum of the lengths of the perpendiculars let fall upon the plane of the base ABC from the angles D, E, F, of the opposite base.

Consequently, denoting the area of the base of the frustum by b^2; the perpendiculars from D, E, F, upon the plane ABC (or the altitudes of the pyramids) by a, a', a'', respectively; and the volume of the frustum by V;

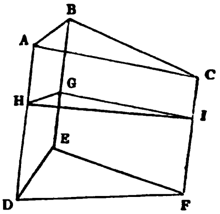
$$V = b^2 \times \frac{a + a' + a''}{3}.$$

If the frustum is upright, the perpendiculars are equal to the lateral edges of the frustum.

a. A plane drawn parallel to one of the bases of a frustum of a triangular prism, at a distance equal to one-third of the sum of the distances of the three opposite solid angles from the plane of this base, determines, with the produced lateral faces of the frustum, a prism which is equal in volume to the frustum; for the volume of each $= b^2 \times \frac{a + a' + a''}{3}$. Each of the lateral edges of this prism is equal to one-third of the sum of the lateral or parallel edges of the frustum; and the volume of this prism is equal to the product of the area of a section made at right angles to its lateral edges by the length of one of its lateral edges (Art. 98. *b.*). Whence, denoting the lengths of the three lateral edges of the frustum by l, l', l'', respectively; the area of the section of the frustum, made at right angles to its lateral edges, by s^2, and the volume by V,

$$V=s^2\times\frac{l+l'+l''}{3}.$$

b. A frustum of a triangular prism is frequently termed a Wedge.



One of the lateral faces of a wedge is termed the back or head of the wedge, and the two remaining lateral faces, the sides.

The straight line formed by the intersection of the two sides is the edge of the wedge.

Let AECF be a wedge: then if AE is the back, AF, BF are the sides, and CF the edge.

In the lateral edge BE, let any point G be taken: in the planes AE, BF, let GH, GI be drawn at right angles to BE, and let HI be joined: the section GHI is at right angles to BE (Planes, Prop. 4.), and consequently to the other lateral edges, AD, CF (Planes, Prop. 7.). If one side of the triangular section, for example $GH=b$, is known, and the perpendicular from I, on $GH=h$ (that is, if the breadth of the back of the wedge, and the perpendicular let fall from a point in the edge, on the plane of the back, are known), the area of the section made at right angles to the lateral edges of the wedge, viz. $\frac{bh}{2}$, is known. Whence, denoting AD, by l; BE by l'; CF, by l''; and the volume of the wedge by V;

$$V=\frac{bh}{2}\times\frac{l+l'+l''}{3}.$$

If the perpendicular from the edge on the plane of the back of the wedge cannot be conveniently measured, let the measured lengths of the three sides of the section be $GI=a$; $GH=b$; $HI=c$: then the area of the section $=\sqrt{\{s(s-a)(s-b)(s-c)\}}$, (Art. 76. *c*).

$$\text{and } V=\sqrt{\{s(s-a)(s-b)(s-c)\}}\times\frac{l+l'+l''}{3}.$$

Examples. 1st. Given $b=5$; $h=8$; $l=12$; $l'=10$; $l''=8$: required V. *Ans.* $V=200$.

CALCULATION.

$$\frac{bh}{2}=\frac{5\times 8}{2}=20:\ \frac{l+l'+l''}{3}=\frac{12+10+8}{3}=10,$$

$$\therefore V=20\times 10=200.$$

2d. Given $a=9$; $b=4$; $c=8$; $l=13$; $l'=14$; $l''=10$: required V.
Ans. $V=197.308\dot{6}$.

CALCULATION.

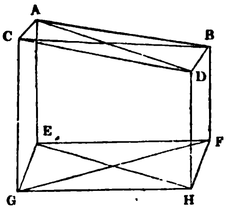
$$s=10.5;\ s-a=1.5;\ s-b=6.5;\ s-c=2.5,$$

$$\sqrt{\{s(s-a)(s-b)(s-c)\}}=\sqrt{\{10.5\times1.5\times6.5\times2.5\}}=\sqrt{255.9375}=15.998,$$

$$\frac{l+l'+l''}{3}=\frac{13+14+10}{3}=\frac{37}{3}=12\tfrac{1}{3}=12.\dot{3}.$$

$$\therefore\ V=15.998\times12.\dot{3}=197.308\dot{6}.$$

104. To find the volume of a frustum of a parallelepiped.



Let EFGHABCD be a frustum of a parallelepiped; let the two diagonal planes AEHD, BFGC be drawn, and let it, in the first instance, be supposed that the lateral edges of the frustum are perpendicular to the plane of the base EFGH.

The plane AEHD divides the frustum ED into the two truncated triangular prisms EFHABD, EGHACD, and the plane BFGC divides it into the two truncated triangular prisms GEFCAB, GHFCBD.

Now $\text{EFHABD}=\frac{1}{3}\text{EFH}(\text{AE}+\text{BF}+\text{DH})$;
$\text{EGHACD}=\frac{1}{3}\text{EGH}(\text{AE}+\text{CG}+\text{DH})$;
$\text{GEFCAB}=\frac{1}{3}\text{GEF}(\text{CG}+\text{AE}+\text{BF})$;
$\text{GHFCBD}=\frac{1}{3}\text{GHF}(\text{CG}+\text{DH}+\text{BF})$.

Taking the sum of these equations, and observing that the four truncated triangular prisms make up twice the frustum ED; also that $\text{EFH}=\text{EGH}=\text{GEF}=\text{GHF}=\frac{1}{2}\text{EFGH}$ (Euc. I. 34.);

$$2\text{ED}=\tfrac{1}{3}\times\tfrac{1}{2}\text{EFGH}(3\text{AE}+3\text{BF}+3\text{CG}+3\text{DH}),$$

$$\text{or } 2\text{ED}=\tfrac{1}{2}\text{EFGH}(\text{AE}+\text{BF}+\text{CG}+\text{DH}),$$

$$\therefore\ \text{ED}=\tfrac{1}{4}\text{EFGH}(\text{AE}+\text{BF}+\text{CG}+\text{DH}).$$

If the lateral edges of ED are not perpendicular to the plane of the base EFGH, it is necessary to replace AE, BF, CG, DH, by the perpendiculars let fall from A, B, C, D, upon the plane EFGH.

Hence the volume of a frustum of a parallelepiped is obtained by multiplying the area of one of the bases of the frustum by one-fourth part of the sum of the four perpendiculars let fall upon this base from the angles of the opposite base: consequently, if the area of the base is denoted by b^2, the perpendiculars by a, a', a'', a''', respectively, and the volume by V,

$$V=b^2\times\frac{a+a'+a''+a'''}{4}.$$

a. A plane drawn parallel to one of the bases of a frustum of a parallelepiped at a distance equal to one-fourth part of the sum of the distances of the four opposite solid angles from the plane of this base determines, with the produced lateral faces of the frustum, a parallelepiped which is equal in volume to the frustum; for the volume of each

$$= b^2 \times \frac{a + a' + a'' + a'''}{4}.$$

b. In the same way it may be proved that the volume of a frustum of any prism whose base is a regular polygon, is given by the equation

$$\text{V} = b^2 \times \frac{a + a' + a'' \ldots\ldots + a^{n-1}}{\text{N}};$$

in which b^2 expresses the area of the base; N, the number of sides of the base; and a, a', a'', &c. the perpendiculars let fall from the angles of the opposite base upon the plane of the base, b^2.

Consequently a frustum of a regular prism is equivalent to a regular prism which has the same base and the same axis.

c. If N is an indefinitely great number, the base of the prism becomes a circle, and the prism a cylinder: whence, also, a frustum of a cylinder is equivalent to a cylinder which has the same base and the same axis.

105. Application of the rule for finding the volume of a wedge, or frustum of a triangular prism, to the determination of the volume of a parapet or rampart.

Let the part of the first figure between the lines A′F′ and WX be the plan of a parapet forming one face of a redoubt on level ground, the crest and foot lines in which the several slopes meet one another being parallel to the horizon; and let the second figure represent a vertical section of the same parapet, taken perpendicularly to the crests.

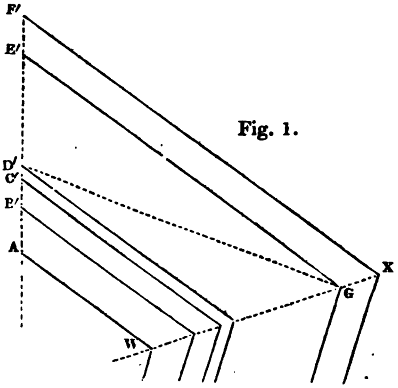


Fig. 1.

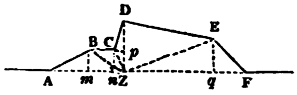


Fig. 2.

Let fall DZ perpendicularly to the ground line AF, and draw the lines BZ, CZ, &c., as in the second figure: then, if planes be imagined to pass through these lines, they will divide the parapet into as many wedges as there are triangles. These last will represent sections perpendicular to the lateral edges of the different wedges; and the face of the redoubt being terminated by vertical planes passing through A′F′ and WX in the first figure, lines equal to those which pass through A′, B′, D′ will be the lateral edges of the wedge whose perpendicular section is ABZ: the lines passing through B′, C′, D′ will be equal to the lateral edges of the wedge whose perpendicular section is BCZ; the line passing through C′ and two lines equal to that which passes through D′ will be the lateral edges of the wedge whose section is CDZ, and so on.

Let A, B, C, &c., represent the areas of the several triangles ABZ, BCZ, CDZ, &c., and let a, b, c, &c., represent the lengths of the several lines passing through A′, B′, C′, &c., on the plane. Then (Art. 103. *b.*)

$$\begin{aligned} &\mathrm{A}\,.\tfrac{1}{3}(a+b+d),\\ &\mathrm{B}\,.\tfrac{1}{3}(b+c+d),\\ &\mathrm{C}\,.\tfrac{1}{3}(c+2d),\\ &\mathrm{D}\,.\tfrac{1}{3}(2d+e),\\ \text{and }&\mathrm{E}\,.\tfrac{1}{3}(d+e+f). \end{aligned}$$

will express the volumes of the several wedges. The sum of these terms may be put in the form

$$\begin{aligned} &\tfrac{1}{3}\mathrm{A}a\\ &+\tfrac{1}{3}(\mathrm{A}+\mathrm{B})b\\ &+\tfrac{1}{3}(\mathrm{B}+\mathrm{C})c\\ &+\tfrac{1}{3}(\mathrm{A}+\mathrm{B}+2\mathrm{C}+2\mathrm{D}+\mathrm{E})d\\ &+\tfrac{1}{3}(\mathrm{D}+\mathrm{E})\,e.\\ &+\tfrac{1}{3}\ \mathrm{E}f. \end{aligned}$$

Hence we obtain the following rule: —

Multiply each horizontal line on the plane by the area of the triangle which, in the section, has an angular point in that line, or by the sum of the areas of all the triangles which have an angular point in that line: one-third of the sum of these products will be the volume of the parapet.

If the sections of the redoubt are alike on all its faces, it is evident that the perimeter of each polygon formed by the crest or foot-lines may be multiplied by the areas of the triangles as above; when one-third of the sum of the products will be the volume of the whole parapet of the redoubt.

In a similar manner may the volume of the ditch of a redoubt be found when its contour lines are parallel to one another.

Example. The dimensions of the parapet on one face of a pentagonal redoubt are as follow:

The horizontal line passing through D = 40 feet.

A′Z = ... 11 ft. 6 in.	whence, by computation,	$a = 23.288$ feet.
ZF′ = ... 20		$b = 32.01$
Bm or Cn = 3		$c = 37.82$
Cp = ... 1 ft. 6 in.		$d = 40.$
Eq = ... 15.		$e = 61.795$
Er = ... 5.		$f = 69.061.$

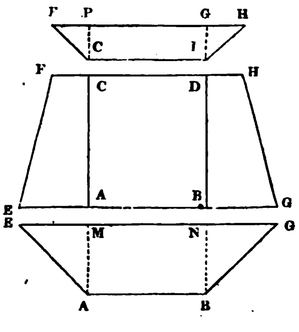
From these data the volume of the parapet is found to be 6493.7916 cubic feet, or 240.5108 cubic yards.

The rule is applicable in whatever manner the dividing lines are drawn on the section; but since, in general, the vertical and horizontal lines of a rampart or parapet are given or may be easily measured, the areas of the triangles in the figure would be more easily computed than those of triangles formed by lines drawn in any other manner.

If the crest and foot-lines in a plan should not be parallel to one another, as when the work is on ground inclined to the horizon, or when it is commanded by heights in the neighbourhood, the parapet or rampart may be divided into wedges by vertical planes imagined to pass through those lines and also diagonally, as shown by D′G, across the slope between every two such lines. The parallel edges of each wedge will then be the vertical lines passing through three points, as D′, E′, G, and the perpendicular section will be the horizontal projection or plan of a triangle, as D′E′G.

In some of these wedges two or one of the vertical lines will become zero; in which cases the volume of the solid will be equal to the product of the horizontal triangle by one-third of the remaining edge or two edges.

106. To find the volume of a prismoidal solid like that which may be conceived to be formed by making an excavation, for a road, on sloping ground. The upper surface of the excavation is supposed to be inclined, but let the lower surface be parallel to the horizon; and, the excavation being terminated by two vertical planes perpendicular to the direction of its length, that surface will be rectangular: also let the sides have any given inclinations.



Let EFHG represent the upper surface of the supposed excavation, projected on a horizontal plane, and let AD be the lower surface: let EABG and FCDH represent the vertical sections at the extremities; and at present let the inclinations of the side planes to the horizon be equal to one another. Draw the vertical lines AM, BN, CP, DG.

If vertical planes pass through AC, BD, and through a line joining C, B, the middle part of the excavation, which is a frustum of a wedge, will be divided into two wedges: then, (Art. 103. *b.*)

$\frac{1}{3}(\text{AM}+\text{NB}+\text{PC}).\frac{1}{2}\text{AB}.\text{AC}$ will be the volume of the wedge whose section perpendicular to the parallel edges is the triangle ABC, and

$\frac{1}{3}(\text{AM}+\text{CP}+\text{DQ}).\frac{1}{2}\text{CD}.\text{DB}$ will be the volume of the wedge whose perpendicular section is CDB. The sum of these volumes is that of the part contained between the vertical planes passing through AC and BD.

Next, if a vertical plane pass through AC, and one through the point E and the line PC, the left-hand part of the excavation will be divided into two wedges, or rather into two pyramids; and we have

$\frac{1}{3}(\text{AM}+\text{PC}).\frac{1}{2}\text{AC}.\text{AE}$ for the volume of the solid of which a section perpendicular to the parallel edges AM, PC, is the triangle EAC; also

$\frac{1}{3}\text{PC}.\frac{1}{2}\text{AC}.\text{FC}$ for the volume of the solid whose section perpendicular to the edge PC is the triangle EFC. The sum of these volumes is that of the solid forming the left-hand side of the excavation.

In a similar manner may the volume of the right-hand side be found; and if the inclinations of the sides of the excavation are equal, the volumes of the two side portions will be equal to one another.

Now let AB, CD be each represented by a,

AM, BN . . . by h,
CP, DQ . . . by k,
EA, BG . . . by b,
FC, DH . . . by d,
and let AC, BD . . . by l.

Substituting these values, the two volumes above found will be expressed by

$$\frac{2h+k}{3}.\frac{al}{2}+\frac{2k+h}{3}.\frac{al}{2};$$

$$\text{and } \frac{h+k}{3}.\frac{bl}{2}+\frac{k}{3}.\frac{ld}{2}.$$

The first, which is the volume of the middle portion, is equivalent to

$$(3ah+3ak).\tfrac{1}{6}l.$$

The second being doubled will express the sum of the two side portions, and will be equivalent to

$$(2hb + 2kb + 2kd).\tfrac{1}{6}l:$$

for $2\,bk$ in this last expression, putting $bk + dh$ its equivalent (since from the similar triangles EMA, FPC we have $b : h :: d : k$; whence $bk = dh$) the expression becomes

$$(2hb + bk + dh + 2kd).\tfrac{1}{6}l:$$

and the sum of the two equivalents is the value of the required volume.

But it is evident that the sum of the terms within the parentheses may be put in the form

$$\begin{aligned} &(b+a)\,h, \\ +&(d+a)\,k, \\ +&(b+d+2\,a)\,(h+k): \end{aligned}$$

the first of these terms is the area of the section EABG, the second is the area of the section FCDH, and the third is four times the area of a vertical section parallel to these, and equally distant from each: therefore the required volume of the whole prismoidal solid is equal to the product of one-sixth of the length by the sum of the areas of the sections at the two extremities, and four times the area of a parallel and equidistant section.

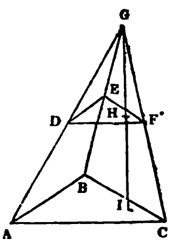
The investigation would be similar to that which has been given if the inclinations of the two sides had been unequal; observing that from the similarity of the triangles EMA, FPC, and BNG, DQH we should have

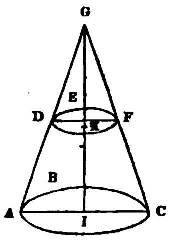
$$\text{EM}\,.\,\text{PC} = \text{FP}\,.\,\text{AM}, \text{ and } \text{BN}\,.\,\text{QH} = \text{NG}\,.\,\text{DQ}:$$

the rule for the volume would be the same: it was first given by Mr. Macneil.

107. To find the volume of a frustum of a pyramid or an upright cone.

Let ABCDEF be a frustum of a pyramid or an upright cone; and let the areas $\text{ABC} = b^2$, $\text{DEF} = t^2$, and the height $\text{HI} = h$, be given to find the volume of the frustum.





Let the lateral planes of the frustum of the pyramid and the sides of the frustum of the cone be produced until they meet in G: also let GHI be perpendicular to the plane ABC, and let GH, which is the altitude of G–DEF, be denoted by x.

Then GI, the altitude of G-ABC $=$ GH $+$ HI $= x + h$.

Since ABC : DEF :: GI^2 : GH^2, (Prisms, &c. Prop. 1. cor.)

$$\text{or,} \quad b^2 : t^2 :: (x+h)^2 : x^2,$$

$$\therefore \quad b : t :: x+h : x, \text{ (Alg. Art. 182. } f.)$$

$$\therefore \quad bx = tx + th, \text{ (Alg. Art. 182.)}$$

$$\therefore \quad bx - tx = th, \text{ and } x = \frac{th}{b-t} = \text{GH},$$

$$\therefore \; x + h = \frac{th}{b-t} + h = \frac{th + bh - th}{b-t} = \frac{bh}{b-t} = \text{GI}.$$

Again, volume G-ABC $= \frac{1}{3}$ABC $\times$ GI $= \frac{1}{3}b^2 \times \frac{bh}{b-t} = \frac{1}{3}h \times \frac{b^3}{b-t}$; (Art. 99. and 101.)

And volume G-DEF $= \frac{1}{3}$DEF $\times$ GH $= \frac{1}{3}t^2 \times \frac{th}{b-t} = \frac{1}{3}h \times \frac{t^3}{b-t}$.

Now, volume ABCDEF $=$ vol. G $-$ ABC $-$ vol. G $-$ DEF.

$$\therefore \text{ vol. ABCDEF} = \tfrac{1}{3}h \times \frac{b^3}{b-t} - \tfrac{1}{3}h \times \frac{t^3}{b-t} = \tfrac{1}{3}h \times \frac{b^3 - t^3}{b-t} = \tfrac{1}{3}h(b^2 + bt + t^2).$$

Since $bt = \sqrt{b^2 t^2}$, bt is a mean proportional (Arith. Art. 345. *a.*) between b^2 and t^2, the areas of the base and top of the frustum.

Whence, to find the volume of a frustum of a pyramid or cone,—Add together the areas of the two ends and the mean proportional between them; multiply this sum by the height, and take one-third of the result for the volume required.

If AF is a frustum of a pyramid, the areas b^2, t^2 are determined, from proper data, by the rules for finding the areas of plane rectilineal figures (Section VIII.). If AF is a frustum of a cone, r, r' being the radii of the base and top, respectively, $b^2 = \pi r^2$; $t^2 = \pi r'^2$ (Art. 86.);

$$bt = \sqrt{\{\pi r^2 \times \pi r'^2\}} = \sqrt{\pi^2 r^2 r'^2} = \pi r r'.$$

Examples. 1st. Given the sides of the base of a frustum of a triangular pyramid equal to 12, 15, and 18 ft.; the sides of the top equal to 8, 10, and 12 ft. respectively; and the height equal to 16 ft.: required the volume of the frustum.

Ans. 1005.381 cubic feet.

CALCULATION.

$$\frac{12+15+18}{2} = \frac{45}{2} = 22.5:$$

$$22.5 - 12 = 10.5; \; 22.5 - 15 = 7.5; \; 22.5 - 18 = 4.5:$$

$$\sqrt{\{22.5 \times 10.5 \times 7.5 \times 4.5\}} = \sqrt{(7973.4375)} = 89.294 = b^2:$$

$$\frac{8+10+12}{2} = \frac{30}{2} = 15:$$

$$15 - 8 = 7; \; 15 - 10 = 5; \; 15 - 12 = 3.$$

$$\sqrt{\{15 \times 7 \times 5 \times 3\}} = \sqrt{(1575)} = 39.686 = t^2,$$

$$\sqrt{\{89.294 \times 39.686\}} = \sqrt{(3543.721684)} = 59.529 = bt,$$

$$\therefore \tfrac{1}{3} h\,(b^2 + bt + t^2) = \tfrac{1}{3} \times 16 \times (89.294 + 59.529 + 39.686) =$$

$$\frac{188.509 \times 16}{3} = \frac{3016.144}{3} = 1005.381 \text{ cubic feet.}$$

2d. Given the radius of the base of a frustum of a cone = 3 feet; the radius of the top = 2 feet; and the height = 12 feet: required the volume of the frustum.

Ans. 238.7616 cubic feet.

CALCULATION.

$3^2 = 9$, and $3.1416 \times 9 = 28.2744$ sq. ft. $= \pi r^2$ or b^2,
$2^2 = 4$, and $3.1416 \times 4 = 12.5664$ sq. ft. $= \pi r'^2$ or t^2,
$3.1416 \times 3 \times 2 = 18.8496$ sq. ft. $= \pi r r'$ or bt.

$$\tfrac{1}{3} \times 12 \times (28.2744 + 18.8496 + 12.5664) = 59.6904 \times 4 = 238.7616 \text{ cubic feet.}$$

3d. Given a side of the base of a frustum of a square pyramid = 25 in.; a side of the top = 9 in.; and the height = 20 ft.: required the volume of the frustum.

Ans. 43.1018 cubic ft.

4th. Given the radius of the base of a frustum of a cone = 9 in.; the radius of the top = $4\frac{1}{2}$ in.; and the height = 14 ft. 3 in.: required the volume of the frustum.

Ans. 14.68943 cubic ft.

5th. Given the circumference of the base of a frustum of a cone = 3 ft.; the circumference of the top = 2 ft. 8 in.; and the height = 21 ft.: required the volume of the frustum.

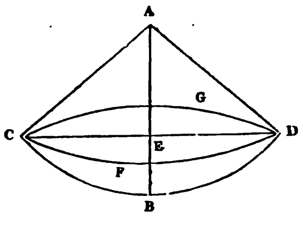
Ans. 13.43083 cubic ft.

6th. The dimensions of a cask, in the form of two conic frustums joined at the bases, are: bung diameter = 2 ft.; head diameter = 1 ft. 8 in.; length = 2 ft. 3 in.: required the capacity of the cask, and the weight of the quantity of water which it is capable of containing, on the supposition that a cubic foot of water weighs 1000 ounces avoirdupois.

Ans. Capacity of cask = 5.9559 cubic feet. Weight of 5.9559 cubic feet of water = 3 cwt. 1 qr. 8 lb. 3.9 oz.

108. To find the volume of a segment of a sphere.

Let ABC be a sector of a circle: from C, let CE be drawn perpendicular to AB; and let ABC revolve about AB: the plane sector ABC produces a spherical sector ABCGDF, the right-angled triangle AEC produces the upright cone ACGDF, and the arc BC produces the spherical segment BCGDF.



Let AB $= r$; BE $= h$; $\therefore$ AE $= r - h$; CE$^2 = h\,(2r - h)$, (Euc. III. 35.)

The volume of sector ABCGDF $= \frac{2}{3}\pi \times \text{AB}^2 \times \text{BE} = \frac{2}{3}\pi r^2 h$. (Cylinder, &c., Prop. 12. Cor. 2.)

And vol. of cone ACGDF $= \frac{\pi \times \text{CE}^2 \times \text{AE}}{3} = \frac{\pi h(2r-h)(r-h)}{3} = \frac{2}{3}\pi r^2 h - \pi r h^2 + \frac{1}{3}\pi h^3$. (Art. 101.)

But vol. of segment BCGDF = vol. of sector ABCGDF — vol. of cone ACGDF,

$\therefore$ vol. of segment BCGDF $= \frac{2}{3}\pi r^2 h - (\frac{2}{3}\pi r^2 h - \pi r h^2 + \frac{1}{3}\pi h^3) = \pi r h^2 - \frac{1}{3}\pi h^3 = \pi h^2\left(r - \frac{h}{3}\right)$.

Hence, to find the volume of a segment of a sphere, when the radius of the sphere and the height of the segment are given:—From the radius of the sphere, subtract a third part of the height of the segment; multiply the remainder by the square of the height of the segment, and this product by the constant π; the result expresses the volume of the proposed segment.

Ex. Given $r = 10$ inches; $h = 4$ inches: required V.
Ans. V $= 435.6352$ cubic inches.

$$r - \frac{h}{3} = 10 - \tfrac{4}{3} = 8\tfrac{2}{3}:\ h^2 = 4^2 = 16;\ \pi = 3.1416.$$

$$\therefore \pi h^2\left(r - \frac{h}{3}\right) = 3.1416 \times 16 \times 8\tfrac{2}{3} = 435.6352 \text{ cubic inches.}$$

a. If CE $(= b)$, the radius of the base, and BE $(= h)$, the height of the segment, are given, to find the volume of the segment:

$$r = \frac{b^2 + h^2}{2h}, \text{ (Art. 96. } a.)$$

$$\therefore \pi h^2\left(r - \frac{h}{3}\right) = \pi h^2\left(\frac{b^2 + h^2}{2h} - \frac{h}{3}\right) = \pi h^2\left(\frac{3b^2 + 3h^2 - 2h^2}{6h}\right) = \tfrac{1}{6}\pi h(3b^2 + h^2).$$

Hence, to find the volume of a segment of a sphere, when the radius of the base of the segment and the height of the segment are given:— To three times the square of the radius of the base of the segment, add the square of the height of the segment; multiply the sum by the height of the segment, and this product by the constant $\frac{\pi}{6}$ $(= .5236)$; the result expresses the volume of the proposed segment.

Examples. 1st. Given the radius of the base of a segment of a sphere, 10 feet; and the height of the segment, 5 feet: required the volume of the segment.
Ans. 850.85 cubic feet.

CALCULATION.

$$\tfrac{1}{6}\pi h = .5236 \times 5 = 2.618:$$
$$3b^2 + h^2 = 3 \times 10^2 + 5^2 = 300 + 25 = 325,$$
$$\therefore \tfrac{1}{6}\pi h(3b^2 + h^2) = 2.618 \times 325 = 850.85 \text{ cubic feet.}$$

2d. Given $r=10.5$; $h=4.5$: required v.
Ans. $v=572.5566$.

3d. Given $b=14$; $h=6.5$: required v.
Ans. $v=2144.99285$.

b. $\frac{1}{6}\pi h\,(3b^2+h^2)=\frac{1}{2}\pi b^2 h+\frac{1}{6}\pi h^3$.

Now, $\frac{1}{2}\pi b^2 h$ expresses half the volume of the cylinder, the radius of whose base is b, and the height, h:

And $\frac{1}{6}\pi h^3$ expresses the volume of the sphere whose diameter is h:

Whence the volume of a segment of a sphere is equal to half the volume of a cylinder which has the same base and the same height; plus the volume of a sphere which has the height of the segment for its diameter.

109. To find the volume of a spherical zone.

A spherical zone being equal to the difference of two segments of a sphere, if the volumes of the segments whose bases are equal, respectively, to the base and top of the zone are found by the formulæ of the last article, and the volume of the less segment is subtracted from that of the greater, the remainder expresses the volume of the proposed zone.

If b, the radius of the base, t, the radius of the top, and h, the height of the zone, are given to find the volume:

$$r\text{, the radius of the sphere}=\tfrac{1}{2}\sqrt{\left\{4t^2+\left(\frac{h^2+b^2-t^2}{h}\right)^2\right\}},$$

(Art. 96. *c.*):

h', the height of the less segment $=r-\sqrt{(r^2-t^2)}$; (see the fig. of Art. 96.)

h'', the height of the greater segment $=r-\sqrt{(r^2-b^2)}$;

And from r, h', h'', the volume of the zone is found as before.

APPENDIX.

1. OF THE PLANE SCALE.

LET any straight line be divided into ten equal parts: let one of the tenth parts be divided into ten equal parts (each of which is consequently $\frac{1}{10}$ of $\frac{1}{10}$ or $\frac{1}{100}$ part of the assumed line): let one of the hundredth parts be divided into ten equal parts, &c.: then, if one part of the last division is assumed as the representative of the unit of length, one foot, for example, two of these parts represent two feet, three represent three feet, &c.: one part of the last division but one represents ten feet; two parts represent twenty feet, &c.: one part of the last division but two represents 100 feet; two parts represent 200 feet, &c.:

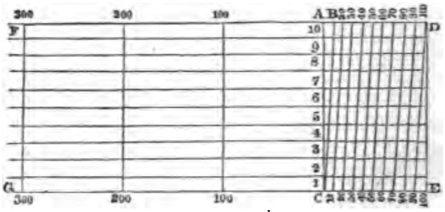
&c. &c. &c.

Or, if the parts of the last division are assumed to represent tenth parts of the unit of length, then the parts of the last division but one represent units of length; the parts of the last division but two represent tens of the unit of length, &c. And, in general, if a line is divided into parts which are equal to tens of each other, or to tenth parts of each other, according as the parts are taken from right to left or from left to right, in whatever division, one part is assumed as the representative of the unit of length, the parts of the successive divisions, if taken towards the left, will express tens, hundreds, thousands, &c.: and if towards the right, tenths, hundredths, thousandths, &c., of the assumed unit.

The construction and use of the scale of equal parts are founded on these principles.

To construct a scale of equal parts, let AB be a straight line which is to form ten equal parts of the scale.

Draw AC at right angles to AB: on AC set off ten equal



parts of any length, from A to C; join CB, and through the points 1, 2, 3, &c., draw parallels to AB: then the parts of the parallels intercepted between CA, CB, will be equal respectively to $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c., of AB, (Euc. VI. 10.).

Let AB and the ten parallels to AB be produced both ways to D and F: and let ten equal distances AB; B, 20; 20, 30; &c., be taken in the straight line AD; and any number A, 100; 100, 200; &c., of distances each equal to AD, from A to F: also let parallels to BC be drawn through the points 20, 30, . . . D, and parallels to AC through the points D; 100; 200; &c. The figure thus constructed is a scale of equal parts, which, from the mode of dividing AB into tenth parts, is generally termed a diagonal scale.

The manner of using this scale is sufficiently simple. As an illustration, let it be required to take 386 from it.

Let one foot of a pair of compasses be placed, in the line GE, at 300, and the other foot at 80; the value of the line intercepted between the feet of the compasses is 380. To augment this line by 6, let the feet of the compasses (the one being kept on the line 300, 300, and the other on the line 80, 90), be moved upwards to the parallel marked 6: the length intercepted between the feet of the compasses is 386. Whether the numerical value of the line is 3860 or 386 or 38.6, &c., the manner of taking the distance from the scale is the same.

The length of a line may be measured by applying a scale of equal parts to the line, or by taking the length of the line in a pair of compasses, and applying the compasses to a scale.

Very short lines, or minute subdivisions of a line, are measured by means of a contrivance named a *Vernier*, for the description of which see Article 66.

If two straight lines are measured with the same scale, the ratio of the numerical values of the lines expresses the ratio of the lines.

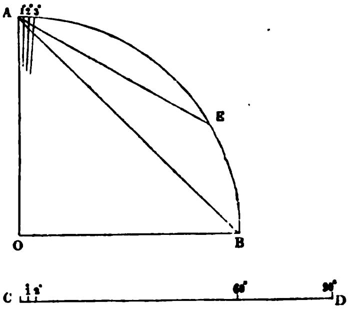
Besides the scale of equal parts, the plane scale of a case of mathematical instruments contains,

1st. A line or scale of chords, marked C.

If AB, the circumference of a quadrant, is divided into 90 equal arcs or degrees, and straight lines are drawn from A to the extremities of the arcs of 1°, 2°, 3°, ... 90°, these lines are the chords of 1°, 2°, 3°, &c., to the radius AO.

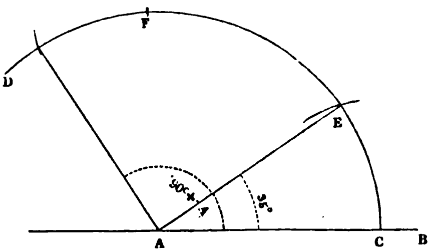
The scale of chords is formed by taking a straight line CD equal to AB, the chord of 90°, or to AE, the chord of 60°, and making C 1° equal to the chord A 1°; C 2° equal to the chord A 2°, &c.

The scale of chords is employed to make an angle of any given number of degrees, or to measure an angle.



For example, at the point A in the straight line AB, let it be required to make an angle of 35°.

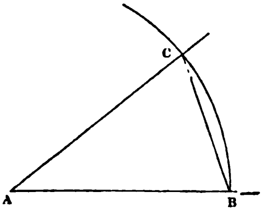
From A, as centre, with the chord of 60°, (which is equal to the radius of the scale, (Euc. IV. 15.) describe a circular arc CD, and from C, as centre, with the chord of 35°, describe another arc, cutting CD in E; join AE. CAE is an angle of 35°.



If the proposed angle is greater than 90°, it will be necessary to divide it into two parts, 90° and A (less than 90°), and to place in the arc CD, first the chord of 90° (CF), and then the chord of A.

To measure an angle with the scale of chords.

Let A be any angle. From A, as centre, with a radius equal to the chord of 60°, describe an arc BC: draw the chord BC, and with a pair of compasses apply the length BC to the scale of chords, measuring from C towards D: the number of degrees intercepted between the feet of the compasses expresses the measure of the angle A.



If the angle A is obtuse, it will be necessary, first, to place in the arc CB, the chord of 90°: and, second, to measure the remaining part of the arc in the manner already explained. Then 90° + the measure of the remaining angle (or arc) expresses the measure of the obtuse angle A.

If the scale of chords should not extend beyond 60°, like processes must be used when the proposed angle, or the angle to be measured, exceeds that number of degrees.

2d. A line or scale of the sines of arcs from 1° to 90°, (see Art. 30. *a.*) which is marked S. or Sin. This scale is used in the orthographic projection of the sphere.

3d. A line of the tangents of arcs from 1° to 90°, (see Art. 30. *b.*); marked T. or Tan.

4th. A line of semi-tangents; marked S. T.

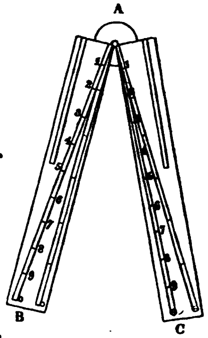
5th. A line of the secants of arcs from 1° to 90°; marked Sec. (Art. 30. *c.*).

The lines 3, 4, 5, are employed in finding the centres and poles of projected circles in the stereographic projection of the sphere.

---

2. OF THE SECTOR.

THE sector is composed of two equal scales or arms, AB, AC, connected at the point A, like the legs of a pair of compasses.



Its use is to resolve questions relative to proportional numbers or lines.

The scales or lines graduated on each arm of the sector are, —

A scale of equal parts (called the line of lines), marked L.

A scale of chords, marked C.

A scale of tangents, marked T.

A scale of sines, marked S.

A scale of secants, marked Sec.

To explain the manner of using the sector, — let AB, AC, be equal scales of equal parts. Then it is evident that if the arms of the sector are separated, and the points 1, 1; 2, 2; 3, 3; &c. are joined, a number of similar isosceles triangles A 1, 1; A 2, 2; &c. are formed, all of which have A for their common vertex.

Supposing now that it is required to find a fourth proportional to three given lines a, b, c.

1st. From the point A take, in the arm AB, a length equal to a, and suppose that its extremity falls on the point 3.

2d. From the point A take, in the arm AB, a length equal to b, and suppose that its extremity falls on the point 5.

3d. Open the instrument until the distance of the two points, marked 3, in the two arms, is equal to c: and suppose $c = 4$.

4th. The distance of the two points marked 5, is a fourth proportional to *a*, *b*, *c*, which, in the instance supposed, is $6\frac{2}{3}$.

The arms are set to the distance *c* with a pair of compasses, and the distance of the two points 5, 5, on the two arms, is taken off with the same instrument, and measured on the scale AB or AC.

The same line of "Lines" may also be employed to divide a given line into any number of equal parts. If, for example, it were required to divide a line into nine equal parts, or to find a line equal to one-ninth of that which is given; the length of this line must be taken between the points of the compasses, and the arms of the sector must be opened till those points fall upon the divisions marked 9 on the "Lines:" then the distance between the points 1 and 1 on the "Lines" will be equal to the part required: the distance from 2 to 2 will be two-ninths, and so on.

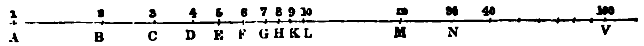
The scales of chords, sines, tangents, &c. on the sector, are used in the construction of problems relating to the projections of the sphere; and the following is an explanation of the manner of taking, from the instrument, the length of the chord, sine, &c. of any number of degrees. The radius of the circle for which the chord, sine, or other trigonometrical line is required, being taken between the points of compasses, the arms of the sector must be opened till the points can be placed on the two brass pins which terminate the lines of chords, sines, &c.: then the distances between the points of division denoting the proposed number of degrees will be the length of the required line.

In general, the scales of chords on the sectors do not extend beyond 60°; the scales of sines extend to 90°; and there are two scales of tangents, one extending from 0 to 45°, and the other from 45° to about 75°. The semi-tangent, as it is called, of any number of degrees, is the tangent of half that number of degrees.

3. OF THE LOGARITHMIC SCALE, OR GUNTER'S SCALE.

THE logarithmic scale, or Gunter's scale, is employed for the same purposes as the sector. The lines usually put on this scale are,—1st, a line of numbers, marked N: 2d, a line of sines, marked S: 3d, a line of tangents, marked T.

The line of numbers is constructed by taking, from a scale of equal parts, the logarithms of the natural numbers from



1 to 10, and setting off each from the beginning of the scale; measuring from A in the direction AV.

The log. of 1 being 0, 1 is written at A, the left extremity of the line.

The log. of 2 is .30103; this distance is taken from a scale of equal parts, and AB made equal to it.

The log. of 3 is .477121; this distance is taken from a scale of equal parts, and AC made equal to it.

And in like manner the logarithms of 4, 5, ... 10, are set off from A to the points marked D, E, ... L. Then AL is a logarithmic scale of numbers from 1 to 10, or from 10 to 100, &c.

Again, if the distance AB be set off from L to M; since AL is the log. of 10, and LM the log. of 2; AM is the log. of 10 + the log. of 2, or the log. of 20.

In like manner the logarithms of 30, 40, ... 100 are formed by setting off from L, in the direction LV, distances equal to AC, AD, ... AL respectively.

If the scale is of sufficient length, the divisions AB, AC, &c. may be divided into parts equal to the lengths of the logarithms of 1.1, 1.2, &c., 2.1, 2.2, &c.

The lines of log. sines and log. tangents are constructed to a radius = 100. The logarithm of the radius is consequently equal to AV or 2.

If $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$, and $\log. a - \log. b = \log. c - \log. d$,

$$\therefore \log. d = \log. c - (\log. a - \log. b).$$

Hence, in any proportion (the terms of which are quantities represented on the scale) the difference, on the scale, between the first and second terms is equal to the difference between the third and fourth terms: and the fourth term is obtained by subtracting, from the third term, the difference between the first and second terms.

The points indicating the log. sin. 90° and log. tan. 45° are placed directly under the point marked 100 on the line of numbers; and, since the nat. sin. and nat. tan. of 34′ 23″ (nearly) are equal to one-hundredth part of the radius, the points which, on the logarithmic scales of sines and tangents, denote those angles correspond to the point marked 1 on the line of numbers.

As an example of the use of the logarithmic scale of sines, in combination with that of numbers, let it be required to find CB from the equation $\text{CB} = \dfrac{\text{AC sin. BAC}}{\text{sin. ABC}}$ (see Art. 71. ex.),

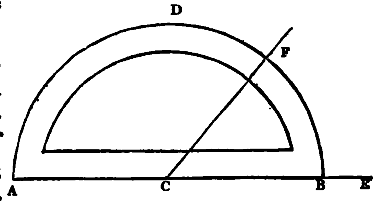
or from the proportion sin. ABC : sin. BAC :: AC : CB.

Extend the points of the compasses from 52° 18′ (= supp. ABC) to 84° 23′ (= BAC), on the line of sines; and, putting one point on 1328 (= AC) on the line of numbers, the other point will extend to 1670 (= CB) on the same line. Thus the required value may be found.

4. OF THE PROTRACTOR.

THE common protractor is a semicircle, the circumference of which is divided into 180 equal parts or degrees.

It is used either to make an angle of any proposed number of degrees, or to measure the magnitude of any angle.

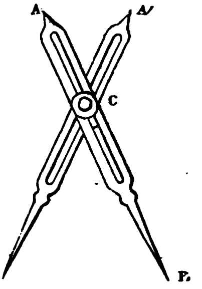


To make at the point C, in the straight line CE, an angle of any number of degrees: Place the centre of the protractor at the point C, and its straight edge (or radius) along the line CE: mark on the paper (suppose at F) the position of the proposed number of degrees on the graduated circumference of the protractor, and draw the straight line CF: the angle ECF contains the proposed number of degrees.

To measure any angle, ECF. Place the centre of the protractor at C, the vertex, and its straight edge along CE, one of the sides of the angle: then the measure of the angle is expressed by the number of degrees marked on the circumference at the point F.

5. OF PROPORTIONAL COMPASSES.

THIS instrument consists of two brass or silver plates, in each of which is a perforation extending nearly its whole length, and each terminates at both ends with steel points. Two short plates, which slide in the perforations, and on each of which is a line serving as an index, are



capable, by means of a screw, C, of being made fast together at any part of the perforation; while the two portions of the compasses are allowed to turn separately on the axis of the screw.

It is manifest that, at whatever point the screw C may be made tight, the legs CA and CA′, CB and CB′ being respectively equal to one another, if the instrument be opened so that the legs form any angle, the distance AA′ will bear the same proportion to BB′ as CA or CA′ bears to CB or CB′.

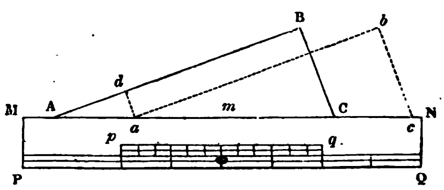
Four scales, of "lines," of "planes," of "solids," and of "polygons," are usually engraven on the instrument: the first serves for dividing a line into equal parts; the points B, B′ being extended so far as to take between them the given line, the distance AA′ will be such a part of that line as is expressed by the reciprocal of the number on the graduation in coincidence with the index at C; or, if the points A, A′ are made to comprehend the given line, the distance BB′ will be the corresponding multiple of that line. It is easy to perceive, therefore, how the instrument may be used for the purposes of enlarging or reducing plans. The lines of planes and solids are but seldom required: if the index near C be set to either of the divisions 2, 3, &c. on the former scale, the distances AA′, BB′ will be to one another in the ratio 1 to $\sqrt{2}$, 1 to $\sqrt{3}$, &c.; and if set to the like divisions on the latter scale, those distances will be to one another as 1 to $2^{\frac{1}{3}}$, 1 to $3^{\frac{1}{3}}$, &c.; the uses of the lines in the constructions of similar planes and similar solid figures must be evident.

The line of "polygons" serves for the inscription of regular polygons in circles; the index at C being placed at the number designating the proposed polygon, and the points B, B′ extended so as to comprehend the radius of the given circle, the distance AA′ will be equal to the side of such polygon.

6. OF MARQUOI'S PARALLEL RULERS.

THESE are instruments of wood, ivory, or metal by which lines parallel and perpendicular to one another, at given distances, may be drawn with considerable facility, and they constitute an improvement on an instrument intended for like purposes, which is said to have been introduced in this country from Germany. They are extensively used in drawing the geometrical planes, profiles, and elevations of works relating to civil and military architecture.

The principal instrument has the form of a triangle, as ABC, right angled at B, of which the hypotenuse AC has any convenient ratio to the shortest side BC, generally that of 3 to 1; and when used for drawing parallel lines at small distances from one



another it is placed, as in the figure, with the hypotenuse in coincidence with an edge of a rectangular ruler, as MQ, 12 inches in length. Along the edge MN, the triangle is made to slide, and it is manifest that if AB be placed in coincidence with a given line, on moving the triangle to any other position, as *abc*, a line drawn along the edge *ab* will be parallel to AB at a distance from it (measured in a direction perpendicular to its length), which bears the same ratio to A*a*, the space described by the angular point A (or by an index *m*), as BC bears to AC: that is, by the construction above supposed, the interval between the parallel lines is equal to $\frac{1}{3}$ of the distance moved along the ruler MQ by the point A, or by the index *m*. The reason is evident, for *ad* being drawn parallel to BC, the triangles A*ad*, ACB are similar to one another; and

A*a* . *ad* :: AC : BC.

The edges MN and PQ, on both faces of the ruler, have scales such as are represented near PQ in the figure: the line *pq* for example is divided so as to form a scale whose smallest graduations are each equal to $\frac{1}{20}$ of an inch, while each of the divisions close to the edge PQ is three times as great, or is equal to $\frac{3}{20}$ of an inch.

If, therefore, it were required to draw by the edge AB of the triangle lines parallel to one another at any distance expressed in twentieths of an inch, each of which may represent a foot, a yard, a fathom, or any other denomination in length (suppose, for example, $\frac{32}{20}$); the hypotenuse AC of the triangle must be placed on the edge PQ, with the index *m* at the zero line in the middle of the scale, and the line AB in coincidence with some given line; then sliding the triangle till the index is in coincidence with the thirty-second division of the scale, a line drawn by the edge AB in the new position will be parallel to the given line, and at a distance from it equal to 32 divisions of the scale *pq*.

It is manifest that if BC were made parallel to any given line, on sliding the triangle on an edge of the ruler to any other position, as *abc*, a line drawn along *bc* will be parallel to the given line, at a distance from it which has, to the space passed over by the index *m*, a ratio expressed by that of 3 to $\sqrt{8}$.

The scales corresponding to *pq* on the two faces of a ruler, as MQ, are usually those of 20, 40, 35, 45 parts to an inch: generally two such rulers are furnished with a triangle; and on the other are scales containing 30, 60, 25, 50 parts in an inch.

When the scale by which a plan is to be drawn is greater than any of those which correspond to *pq* on the rulers, one of the scales, as PQ, on an edge of the ruler, may be used; in this case, if lines parallel to one which is given are to be drawn along the edge AB of the triangle, at distances equal to any given number of divisions on the latter scale, it is evident that after bringing AB to the given line with the index *m* at the zero of the scale, that index must be moved over three times the number of divisions. When lines are to be drawn parallel to one another at considerable distances, the edge BC of the triangle may be placed contiguously to an edge as MN of a ruler; and the triangle being made to slide on that edge as far, if necessary, as the whole length of the ruler, lines drawn along the edge AB will be parallel to one another.

Marquoi's rulers are also used with great convenience in drawing lines perpendicular to a given line: thus let the triangle ABC, with MQ applied to it as in the figure, be put in such a position that BC may coincide with the given line; then, moving the hypotenuse AC along MN, all the lines drawn along the edge AB will be perpendicular to the given line.

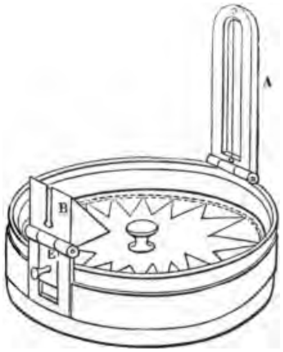
In some cases a ruler having on one only of its edges a scale consisting of divisions equal to $\frac{1}{10}$ of an inch is used; and, in order to be enabled to draw lines parallel to one another, at various intervals smaller than the extent of one of these divisions, several right-angled triangles having sides in different ratios to the hypotenuses are provided. Thus, if the ratio of one side, as AB, to the hypotenuse, be that of 1 to 2, the interval between the lines parallel to AB will be $\frac{1}{20}$ of an inch; if the ratio be as 1 to 3, the interval will be $\frac{1}{30}$ of an inch, and so on.

Keith's parallel rulers are nearly similar to those of Marquoi; but the ratio of the hypotenuse to a perpendicular let fall upon it from the opposite angle is as 4 to 1.

7. OF THE SURVEYING COMPASS.

THE surveying compass consists of a magnetised needle supported on a pivot in the centre of a cylindrical box about 3 inches in diameter and 1 inch in depth, and capable of turning freely round in a horizontal position.

Attached to the upper surface of the needle, and carried round with it, is a circular card, the diameter of which is a little less than that of the box. The circumference of the card is divided into half degrees, beginning from a point over the north pole of the needle. In some instruments the degrees are numbered from 0 to 360°; in others from 0, both ways to 180°; in others from 0 to 180°, and again, in continuation, from 1° to 180°.



To the convex surface of the box are attached, 1st, a sight vane, A, having a longitudinal opening, along the middle of which is stretched a fine thread or wire; and, 2d, diametrically opposite to A, a second sight vane, B, with the interior surface of which is connected a triangular prismatic lens, which can be raised or depressed by a slide at E.

The eye being directed through the opening B to the thread A, the thread and that division on the circumference of the compass card which is at the foot of the thread are made, by the prismatic lens, to appear in the same straight line. By this arrangement, the determination of the number which is nearest to the foot of the thread is rendered easy, and the number can be read without moving the eye from B, or losing the coincidence of the thread A with a distant object.

One extremity of the needle always points to the magnetic north, and consequently the diameter passing through the zero of the compass card coincides in direction with the magnetic meridian. Hence, if the diameter passing through the sight B and the thread A, is made to coincide with the direction of the visual ray, from a distant object to the eye of an observer, the division on the card, which coincides with the thread A, gives the measure of the angle contained between the magnetic meridian and the direction of the distant object. This angle is termed the *Azimuth*, or the *Bearing* of the object.

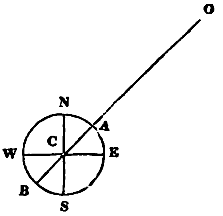
The magnetic, does not coincide with the true meridian. The angle between them, which is termed the variation or declination of the compass, varies at different times and in different places. At Woolwich, in July, 1843, it was 23° 11′ North-west.

Magnetic meridians near each other are assumed to be parallel to each other.

The sight vanes, A, B, are provided with hinges, by means of which they are folded down over the compass card. When the instrument is to be used they are turned up so as to be at right angles to the plane of the card: and by moving the slide E, the prism B is raised or lowered until the thread A and the numbers on the edge of the compass card can be distinctly seen. The instrument is either held in the hand, or mounted on a triangular stand.

To take the bearing of a distant object.

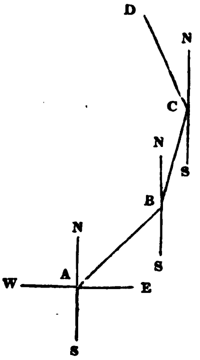
Let O be a spire whose bearing from C is to be taken.



The centre of the compass being placed at C, and the eye at the sight B, the instrument is turned round until the vertical thread A bisects the spire O. Then, supposing that the angle NCA is 54°, the bearing of the spire is 54° from the magnetic north. If E, W, are the east and west points, the bearing CO is 54° N. E., or 54° from the magnetic north towards the east.

The plan of a road, or of the boundary of a field, or a wood, &c., may be made by beginning at any angle A, taking the bearing and length of AB, the straight part of the road or boundary from the angle A to the next angle B; taking the bearing and length of BC, and proceeding in this manner till the bearings and lengths of all the straight lines which form the crooked line ABCDE . . . are taken; and constructing a similar figure on paper.

For example, yards.
if the bearing of AB is 45° N.E., and the length of AB = 150
the bearing of BC is 15° N.E., and the length of BC = 100
the bearing of CD is 24° N.W., and the length of CD = 180
&c. &c. &c.



and A is the point from which the plan is to be traced: through A, draw the magnetic meridian AN, and WE perpendicular to AN, to mark the directions east and west: on the E side of AN make an angle NAB = 45°; and make AB = 150 yds.: through B draw BN parallel to AN (BN is the magnetic meridian passing through B): on the E side of BN make an angle NBC = 15°, and make BC = 100 yds.: through C draw CN parallel to AN or BN, and on the west side of CN make an angle NCD = 24°, &c.

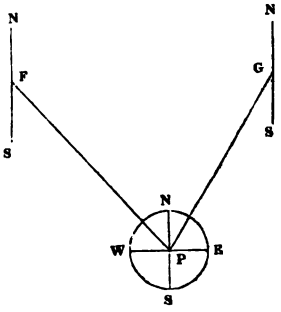
ABCD . . is a plan of the road or boundary.

If the angles which the adjacent sides make with each other are required,

$$ABC = ABN + NBC = (180^\circ - NAB) + NBC = 180^\circ - 45^\circ + 15^\circ = 150^\circ.$$
$$BCD = 180^\circ - BCS - DCN = 180^\circ - NBC - DCN = 180^\circ - 15^\circ - 24^\circ = 141^\circ.$$

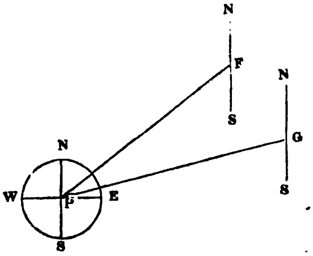
If the figure is a polygon, the positions of the beginning and end ought to coincide. Also, the sum of all the angles, together with four right angles, ought to be equal to twice as many right angles as the polygon has sides (Euc. I. 32. cor.).

To find the angle contained by two straight lines conceived to be drawn from a point to two distant objects.



Let P be the point, and F, G, the distant objects.

Let the bearings NPF, NPG, be measured; then if the directions PF, PG lie on opposite sides of the meridian NPS, the sum of the angles NPF, NPG is the angle required: but if on the same side of the meridian NPS, the difference of the angles NPF, NPG, is the angle required.

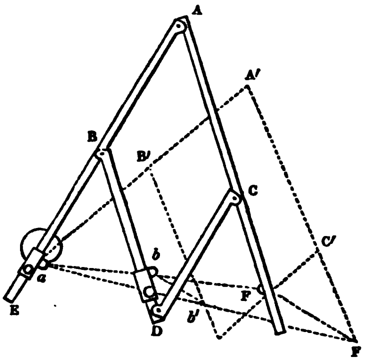


If F, G, are two given points in a plan, the place of P in the plan may be found by means of the two bearings NPF, NPG. For if magnetic meridians are drawn through the points F, G, the alternate angles NPF, PFS are equal; and also the alternate angles NPG, PGS. Whence if, at the point F, the angle SFP is made equal to NPF, and at the point G, the angle SGP is made equal to NPG, the place of P is determined by the intersection of the lines FP, GP.

It will be observed that the whole use of the compass, in these processes, is to measure angles.

8. OF THE PANTAGRAPH.

THIS instrument, which is extensively used for the purpose of enlarging or reducing maps or plans, consists of four bars of brass which are capable of turning freely on joints or axes at A, B, C, and D; these are the angular points of a parallelogram which is formed by the bars when in all possible situations. A frame carrying a short



tube in a vertical position is made to slide on BE, the prolongation of AB; a similar frame with its tube slides on the bar BD, and a tube is fixed at F in the prolongation of AC.

A wire fixed vertically in a cylinder of lead enters, as at *a*, into the tube on BE, and serves as a fulcrum upon which the whole instrument turns; the tube at F and that which slides on BD carries, one of them, suppose at *b*, a pencil, and the other a pointed wire or tracing point. The bars BE and BD are graduated, and, when the frames carrying the tubes are in their proper situations, the points *a*, *b*, and F are in a straight line: from the construction of the instrument this rectilinear direction of the points is retained, whatever be its position about the fulcrum *a*.

In the diagram the points *a* and *b* are supposed to be placed on their respective arms at the graduation marked 1 — 2, in which situations

$$ab : a\text{F} \;(:: a\text{B} : a\text{A}) :: 1 : 2.$$

Now, let the instrument be turned about *a* till it takes some other position, as $a\,\text{A}'\text{F}'$: then, since $a\text{B}'$, $a\text{A}'$ are identical with $a\text{B}$, $a\text{A}$, we have

$$ab' : a\text{F}' :: 1 : 2,$$

and consequently,

$$bb' : \text{FF}' :: 1 : 2.$$

Hence it is evident that if the tracing point at F be made to pass over the lines on any given map or plan, a pencil at *b* will describe a similar figure in which the several lines will be equal in length to one half of the corresponding lines in the given figure. If the tracing point were placed at *b* and the pencil at F, it is evident also that, on the former being made to pass over the lines on a given plan, the latter would describe a similar figure in which each line would be twice the length of the corresponding lines in the given figure.

In like manner, by placing the frames *a* and *b* at other graduations on the arms BE and BD, maps or plans may be enlarged or reduced in any proportion between the homologous lines, according to the ratio between the distances of the fulcrum *a* from B and A.

If the fulcrum be placed at *b*, a pencil at *a*, and the tracing point at F, a copy of an original map will be made, of equal size, but in a reversed position.

9. SOLUTION OF A CUBIC EQUATION BY THE ARITHMETIC OF SINES.

It is proved in Art. 41. that $\cos. 3\text{A} = 4\cos.^3\text{A} - 3\cos.\text{A}$; or, writing $\frac{a}{3}$ for A, that $\cos. a = 4\cos.^3\frac{a}{3} - 3\cos.\frac{a}{3}$,

$$\therefore \cos.^3\frac{a}{3} - \frac{3}{4}\cos.\frac{a}{3} = \frac{1}{4}\cos. a \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

Assuming $\cos.\frac{a}{3}=\frac{x}{n}\left(\text{or } x=n\cos.\frac{a}{3}\right)$, and substituting this expression in equation 1.,

$$\frac{x^3}{n^3}-\frac{3}{4}\frac{x}{n}=\frac{1}{4}\cos. a,$$

$$\text{or, } x^3-\frac{3n^2}{4}x=n^3\times\frac{1}{4}\cos. a \quad . \quad . \quad . \quad . \quad . \quad . \quad 2$$

Now, $n\cos.\frac{a}{3}$ is evidently one of the roots of equation 2.: and since $\cos. a$, $\cos.(360°-a)$, $\cos.(360°+a)$ are all expressed by the same quantity (Art. 24. *b.*) $n\cos.\left(\frac{360°-a}{3}\right)$, $n\cos.\left(\frac{360°+a}{3}\right)$, are also roots of equation 2.

But $\cos.\frac{360°-a}{3}=\sin.\left(90-\frac{360°-a}{3}\right)$ (Art. 17.) $=\sin.\left(-\frac{90°-a}{3}\right)=-\sin.\frac{90°-a}{3}$ (Art. 19. *e.*)

$$\text{And, similarly, } \cos.\frac{360°+a}{3}=-\sin.\frac{90°+a}{3};$$

Whence, $n\times\frac{\cos. a}{3}$; $-n\times\sin.\frac{90°-a}{3}$; $-n\times\sin.\frac{90°+a}{3}$ are the roots of equation 2.

If the roots of the equation $x^3-px=q$ are required; comparing this equation with the equation $x^3-\frac{3n^2}{4}x=n^3\times\frac{1}{4}\cos. a$, it is evident that if the quantities n and $\cos. a$ are assumed such that $\frac{3n^2}{4}=p$, and $n^3\times\frac{1}{4}\cos. a=q$, the two equations are rendered identical, and their roots are consequently expressed by the same quantities.

$$\text{But if } \frac{3n^2}{4}=p,\ n^2=\frac{4p}{3},\ \text{and } n=\frac{2\sqrt{p}}{\sqrt{3}},$$

$$\text{And if } n^3\times\tfrac{1}{4}\cos. a=q,\ \cos. a=4q\div n^3=4q\div\frac{8\sqrt{p^3}}{\sqrt{27}}=\frac{\sqrt{27q^2}}{\sqrt{4p^3}}=\frac{3q\sqrt{3}}{2p\sqrt{p}}.$$

As the greatest value of $\cos. a$ cannot exceed 1, it follows that $\frac{3q\sqrt{3}}{2p\sqrt{p}}$ is a proper fraction, and, consequently, that $2p\sqrt{p}>3q\sqrt{3}$, or that $4p^3>27q^2$ or $\frac{1}{27}p^3>\frac{1}{4}q^2$.

The quantity p being negative, the proposed equation $(x^3-px=q)$ falls under the second case of Art. 289. Alg., which is termed the irreducible case. This case can be, therefore resolved with the help of the trigonometrical tables.

For $x^3-px=q$, being the general form of the equation, deprived of its second term, if in the trigonometrical tables an arc a is found such that

$$\text{natural } \cos. a=\frac{3q\sqrt{3}}{2p\sqrt{p}}, \text{ the roots of the equation are}$$

$$x=\quad n\times\cos.\frac{a}{3}=\frac{2\sqrt{p}}{\sqrt{3}}\times\cos.\frac{a}{3};$$

$$x = -n \times \sin.\frac{90^\circ - a}{3} = -\frac{2\sqrt{p}}{\sqrt{3}} \times \sin.\frac{90^\circ - a}{3};$$

$$x = -n \times \sin.\frac{90^\circ + a}{3} = -\frac{2\sqrt{p}}{\sqrt{3}} \times \sin.\frac{90^\circ + a}{3}.$$

If attention is given to the sign of q, these formulæ may be employed whether the equation is $x^3 - px = q$, or $x^3 - px = -q$.

Otherwise, a solution in the case of q negative can be obtained from the formula

$$\sin.^3\frac{a}{3} - \frac{3}{4}\sin.\frac{a}{3} = -\sin.\,a,$$

in the same manner as the preceding solution, in the case of q positive, has been deduced from the formula

$$\cos.^3\frac{a}{3} - \frac{3}{4}\cos.\frac{a}{3} = \frac{1}{4}\cos.\,a.$$

$$\text{The roots are } x = \frac{2\sqrt{p}}{\sqrt{3}} \times \sin.\frac{a}{3};$$

$$x = \frac{2\sqrt{p}}{\sqrt{3}} \times \cos.\frac{90^\circ + a}{3};$$

$$x = -\frac{2\sqrt{p}}{\sqrt{3}} \times \cos.\frac{90^\circ - a}{3}.$$

Ex. Required the roots of the equation $x^3 - 3x = 1$.

In this instance, $p = 3$; $q = 1$,

$$\therefore \frac{3q\sqrt{3}}{2p\sqrt{p}} = \frac{3 \times \sqrt{3}}{2 \times 3\sqrt{3}} = \frac{1}{2} = \cos.\,60^\circ = \cos.\,a;$$

$$\therefore \cos.\frac{a}{3} = \cos.\,20^\circ; \ \sin.\frac{90^\circ - a}{3} = \sin.\,10^\circ; \ \sin.\frac{90^\circ + a}{3} = \sin.\,50^\circ:$$

$$\text{also, } \frac{2\sqrt{p}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2;$$

$$\begin{aligned} \therefore x &= 2\cos.\,20^\circ = 2 \times .9396926 = 1.8793852; \\ x &= -2\sin.\,10^\circ = -2 \times .1736482 = -.3472964; \\ x &= -2\sin.\,50^\circ = -2 \times .7660444 = -1.5320888. \end{aligned}$$

10. LOGARITHMIC SERIES FOR a^x, ε^x, ε^{-x}, log. $(1+x)$, &c.

LET a be a constant quantity, x a variable, and A, B, C, . . . indeterminate coefficients. Then, if a^x is expanded into a series arranged according to the ascending powers of x, the first term of the developement ought to be 1; because, if x is made equal to 0, a^x becomes $a^0 = 1$ (Alg. Art. 36.), and all the terms of the second member of the series, which are multiplied by x, vanish. Let, therefore,

$$a^x = 1 + \mathrm{A}x + \mathrm{B}x^2 + \mathrm{C}x^3 + \mathrm{D}x^4 + \&c. \quad . \quad . \quad . \quad . \quad . \quad 1$$

Changing the variable x into y, let

$$a^y = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c. \quad . \quad . \quad . \quad . \quad . \qquad 2$$

Next, changing the variable x into $x + y$, let

$$a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + D(x+y)^4 + \&c.. \qquad 3$$

Multiplying the series

$$a^x = 1 + Ax + Bx^2 + \&c. \text{ by the series } a^y = 1 + Ay + By^2 + \&c.$$

$$a^x \times a^y \text{ or } a^{x+y} = (1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.)\,(1 + Ay + By^2 + Cy^3 + Dy^4 + \&c.) \quad . \quad . \quad . \quad . \qquad 4.$$

Since the first members of equations 3, 4, are equal, the second members are equal; and since the equations are true whatever be the value of y, the coefficients of the same powers of y must be also equal (Alg. Art. 174.).

Now the coefficient of the first power of y in eq. 3. is $A + 2Bx + 3Cx^2 + 4Dx^3 + \&c.$

And the coefficient of the first power of y in eq. 4. is $A + A^2x + ABx^2 + ACx^3 + \&c.$

Consequently, $A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. = A + A^2x + ABx^2 + ACx^3 + \&c.$ 5

These expressions being equal, whatever be the value of x, the coefficients of the same powers of x in equation 5. are also equal. Whence

$$2B = A^2; \; 3C = AB; \; 4D = AC, \&c.$$

$$\therefore B = \frac{A^2}{2}; \quad C = \frac{A^3}{2.3}; \quad D = \frac{A^4}{2.3.4} \&c.$$

Substituting these values of B, C, D, &c., in equation 1.,

$$a^x = 1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{2.3}x^3 + \frac{A^4}{2.3.4}x^4 + \&c. \quad . \quad . \quad . \qquad 6$$

To determine the quantity A: let $Ax = 1$, or $x = \frac{1}{A}$, and let $\frac{1}{A}$ be substituted for x, $\frac{1}{A^2}$ for x^2, $\frac{1}{A^3}$ for x^3, &c., in equation 6

$$\therefore a^{\frac{1}{A}} = 1 + \frac{A}{1}.\frac{1}{A} + \frac{A^2}{1.2}.\frac{1}{A^2} + \frac{A^3}{1.2.3}.\frac{1}{A^3} + \frac{A^4}{1.2.3.4}.\frac{1}{A^4} + \&c.$$

$$\text{or, } a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \quad . \quad . \qquad 7$$

Denoting the numerical series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c.$ by ε, equation 7. becomes

$$a^{\frac{1}{A}} = \varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad 8$$

Taking the logarithms of both members of equation 8.,

$$\log.\,(a^{\frac{1}{A}}) = \log.\,\varepsilon, \text{ or } \frac{1}{A}\log.\,a = \log.\,\varepsilon,$$

$$\therefore \log.\,a = A\log.\,\varepsilon, \text{ and } A = \frac{\log.\,a}{\log.\,\varepsilon}.$$

The number ε (the value of which, computed from the numerical series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c.$, is 2.718281828 +) is the base of Napier's system of logarithms. Denoting the logarithms of this system by the symbol l_{ϵ}, and supposing that the logarithmic values of both members of equation 8 are taken in Napier's system, the logarithm of ϵ becomes 1 and $\text{A} = l_{\epsilon} a$. Substituting this value of A in equation 6,

$$a^x = 1 + \frac{l_{\epsilon} a . x}{1} + \frac{(l_{\epsilon} a)^2 x^2}{1.2} + \frac{(l_{\epsilon} a)^3 x^3}{1.2.3} + \text{\&c.} \quad . \quad 9$$

To change the exponential equation $a^x = 1 + \frac{l_{\epsilon} a . x}{1} + \text{\&c.}$ into the exponential equation for ϵ^x, it is only necessary to change a into ϵ and $l_{\epsilon} a$ into $l_{\epsilon} \epsilon$ or 1. With these changes, equation 9. becomes

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{\&c.} \quad . \quad . \quad 10$$

If x is changed into $-x$, the even powers of x are +, and the odd powers −,

$$\therefore \epsilon^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \text{\&c.} \quad . \quad . \quad . \quad 10'$$

a. Since log. $1 = 0$, it is evident that log. $(1 + x)$ becomes zero when $x = 0$.

The indeterminate series proper to represent log. $(1 + x)$ cannot therefore be of the form

$$\log.(1 + x) = \text{A} + \text{B}x + \text{C}x^2 + \text{\&c.}$$

Assume, therefore, $\log.(1 + x) = \text{A}x + \text{B}x^2 + \text{C}x^3 + \text{\&c.} \quad . \quad . \quad . \quad . \quad 1$

x being, in this expression, a variable quantity, let x become $x + y$;

then $\log.(1+x+y) = \text{A}(x+y) + \text{B}(x+y)^2 + \text{C}(x+y)^3 + \text{\&c.} \quad . \quad . \quad . \quad 2$

But $1 + x + y = (1 + x)\left(\frac{1 + x + y}{1 + x}\right) = (1 + x)\left(1 + \frac{y}{1 + x}\right)$,

$$\therefore \log.(1 + x + y) = \log.(1 + x) + \log.\left(1 + \frac{y}{1 + x}\right) \quad . \quad . \quad . \quad . \quad 3$$

Now, if in equation 1., the variable x is changed into $\frac{y}{1 + x}$,

$$\log.\left(1 + \frac{y}{1 + x}\right) = \frac{\text{A}y}{1 + x} + \frac{\text{B}y^2}{(1 + x)^2} + \frac{\text{C}y^3}{(1 + x)^3} + \text{\&c.}$$

Substituting this value of $\log.\left(1 + \frac{y}{1 + x}\right)$ in equation 3,

$$\log.(1 + x + y) = \log.(1 + x) + \frac{\text{A}y}{1 + x} + \frac{\text{B}y^2}{(1 + x)^2} + \frac{\text{C}y^3}{(1 + x)^3} + \text{\&c.} \ldots 4$$

Equations 2. and 4. being true for every value of the variable y, the coefficients of the same powers of y, in both equations must be equal. Whence

$$\text{A} + 2\text{B}x + 3\text{C}x^2 + 4\text{D}x^3 + \text{\&c.} = \frac{\text{A}}{1 + x}.$$

Multiplying both members by $1 + x$,

$$\begin{array}{l|l|l|l} \text{A} + \text{A} & x + 2\text{B} & x^2 + 3\text{C} & x^3 + \text{\&c.} = \text{A}, \\ \phantom{\text{A}} + 2\text{B} & + 3\text{C} & + 4\text{D} & \end{array}$$

$$\therefore (\text{A} + 2\text{B})x + (2\text{B} + 3\text{C})x^2 + (3\text{C} + 4\text{D})x^3 + \text{\&c.} = \text{A} - \text{A} = 0,$$

$$\therefore \text{A} + 2\text{B} + (2\text{B} + 3\text{C})x + (3\text{C} + 4\text{D})x^2 + \text{\&c.} = 0.$$

If, in this series, the variable $x=0$, $A+2B=0 \therefore B=-\frac{1}{2}A$. Similarly,

it is found that $2B+3C=0$, and that $C=-\frac{2}{3}B=-\frac{2}{3}$ of $-\frac{1}{2}A=\ \frac{1}{3}A$;
that $3C+4D=0$, and that $D=-\frac{3}{4}C=-\frac{3}{4}$ of $\frac{1}{3}A=-\frac{1}{4}A$;

Substituting these values of B, C, D, &c., in equation 1.,

$$\log.(1+x)=A\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.\right)\quad .\ .\ .\quad 5$$

If both members of eq. 5. are divided by x,

$$\frac{\log.(1+x)}{x}=A\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\&c.\right).$$

If, to determine A, from this expression, x is made equal to 0, the result obtained is

$$\frac{0}{0}=A, \text{ which is indeterminate.}$$

Let $x=\frac{1}{z}$, and therefore $\frac{\log.(1+x)}{x}=\log.\left(1+\frac{1}{z}\right)\div\frac{1}{z}=z\log.\left(1+\frac{1}{z}\right)=\log.\left(1+\frac{1}{z}\right)^z$.

Expanding $\left(1+\frac{1}{z}\right)^z$ into a series by the binomial formula (Alg. Art. 175.)

$$\begin{aligned}\left(1+\frac{1}{z}\right)^z&=1+\frac{z}{1}\left(\frac{1}{z}\right)+\frac{z}{1}\cdot\frac{z-1}{2}\left(\frac{1}{z}\right)^2+\frac{z}{1}\cdot\frac{z-1}{2}\cdot\frac{z-2}{3}\left(\frac{1}{z}\right)^3+\&c.\\&=1+\quad 1\quad+\quad\frac{z-1}{2}\cdot\frac{1}{z}\quad+\quad\frac{z-1}{2}\cdot\frac{z-2}{3}\left(\frac{1}{z}\right)^2+\&c.\\&=\quad 2\quad+\quad\left(\frac{1}{2}-\frac{1}{2z}\right)+\left(\frac{1}{2}-\frac{1}{2z}\right)\left(\frac{1}{3}-\frac{2}{3z}\right)+\left(\frac{1}{2}-\frac{1}{2z}\right)\left(\frac{1}{3}+\frac{2}{3z}\right)\left(\frac{1}{4}-\frac{3}{4z}\right).\end{aligned}$$

Now, if $x=\frac{1}{z}=0$, $z=\infty$: but if $z=\infty$, the quantities $\frac{1}{2z}, \frac{2}{3z}, \frac{3}{4z}$, &c. are severally equal to zero, and the second member of the last equation is reduced to

$$2+\frac{1}{2}+\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}+\&c.$$

$$\text{or to } 2+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+\&c.$$

This numerical series has been already denoted by the character ε,

Under these conditions, therefore, $\log.\left(1+\frac{1}{z}\right)^z=A=\log.\varepsilon$.

Substituting $\log.\varepsilon$ for A in equation 5,

$$\log.(1+x)=\log.\varepsilon\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.\right)$$

Taking the logarithms in Napier's system, in which $\log.\varepsilon=1$,

$$l_{,}(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.\quad .\ .\ .\quad 6$$

If x is changed into $-x$, this expression becomes

$$l_{,}(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\&c.\quad .\ .\ .\quad 7$$

Now, if $l_\iota(1-x)$ is subtracted from $l_\iota(1+x)$ the remainder is the logarithm of the quotient of $1+x$ divided by $1-x$, or of $\frac{1+x}{1-x}$,

$$\therefore l_\iota\left(\frac{1+x}{1-x}\right)=2\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\&c.\right) \quad . \quad . \quad . \quad 8$$

b. If $1+\frac{z}{n}$ or $\frac{n+z}{n}$ is substituted for $\frac{1+x}{1-x}$, and therefore $\frac{z}{2n+z}$ for x, $\left(\frac{z}{2n+z}\right)^2$ for x^2, &c., in the last formula,

$$l_\iota\left(\frac{n+z}{n}\right)=2\left\{\frac{z}{2n+z}+\frac{1}{3}\left(\frac{z}{2n+z}\right)^3+\frac{1}{5}\left(\frac{z}{2n+z}\right)^5+\&c.\right\}.$$

But $l_\iota\left(\frac{n+z}{n}\right)=l_\iota(n+z)-l_\iota n$: whence, substituting this expression for its equivalent, and transposing $l_\iota n$ to the 2d member of the equation,

$$l_\iota(n+z)=l_\iota n+2\left\{\frac{z}{2n+z}+\frac{1}{3}\left(\frac{z}{2n+z}\right)^3+\&c.\right\}.$$

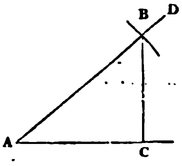
Napier's logarithm of n being known, this formula gives the log. of $n+z$.

11. OF THE RESOLUTION OF TRIANGLES BY CONSTRUCTION.

IN any triangle, if the numerical values of two angles and a side, or of two sides and an angle opposite one of these sides, or of two sides and the included angle, or of the three sides, are given, and a triangle is constructed with the representatives of these values from a scale of equal parts, and a scale of chords or a protractor, the sides of the proposed and of the constructed triangle are proportional to each other, and the two triangles are equiangular and similar (Euc. VI. 4, 5, 6, 7.). Consequently approximations to the values of the unknown parts of the proposed triangle, whether sides or angles, may be obtained by measuring the corresponding sides and angles of the constructed triangle with the scales employed in its construction.

Taking as examples,

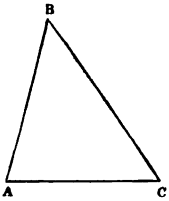
1st. The first example of Art. 58. Case I., in which the hypotenuse $c = 361.4$ yds., and the acute angle $A = 41° 12'$, of a right-angled triangle are given to find a, b.



Construction.—Draw the indefinite line AD: take, from a scale of equal parts, the distance 361.4, and from AD cut off AB equal to this distance (Euc. I. 3.): at the point A, in the straight line AB, make the angle BAE $=41° 12'$ (Art. 1. and 4. App.), and from B let fall the perpendicular BC upon (Euc. I. 12.). Then ABC is the triangle required: and if AC, BC are measured on the scale from which the distance AB was taken, it will be found that AC or $b = 272$ yds. nearly, and BC or $a = 238$ yds.

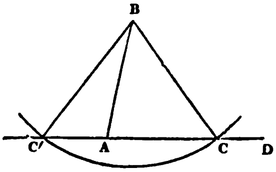
2d. Art. 59., Case I., Ex. 1., in which $A = 74^\circ\ 14'$, $B = 49^\circ\ 23'$, $a = 471.5$, are given to find C, b, c.

Construction.—From a scale of equal parts take $BC = 471.5$: at the point B in the straight line BC make the angle $CBA = 49^\circ\ 23'$, and at the point C in the straight line BC make the angle $BCA = 180^\circ - (74^\circ\ 14' + 49^\circ\ 23') = 56^\circ\ 23'$. ABC is the triangle required; and if AC, AB are measured with the proper scale of equal parts, it will be found that $AC = 372$ nearly, and $AB = 408$.



3d. Art. 59., Case II., Ex. 1., in which $A = 78^\circ\ 13'$, $a = 153$, $c = 137$, are given to find B, C, b.

Construction. Draw the indefinite line AD: at the point A make an angle $DAB = 78^\circ\ 13'$; make $AB = 137$, and from B as centre with a radius $= 153$, describe the circumference of a circle cutting AD in C: join BC: then ABC is the triangle required; and if AC is measured with a scale of equal parts, and the angles B, C with a scale of chords or a protractor, the following values are found, viz.



$$b = 101.6;\quad B = 40^\circ\tfrac{1}{2};\quad C = 61^\circ\tfrac{1}{4}, \text{ nearly.}$$

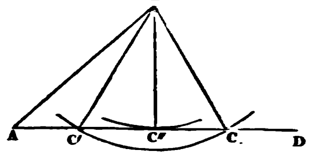
Since BA is less than BC, the point A falls within the circumference of the circle described from the centre B with the radius BC: the two points c, c', in which the circumference cuts AC, are, consequently, on opposite sides of the point A. Let BC′ be joined.

Then, in the triangle BAC′, the sides BA, BC′ are equal to BA, BC, respectively; but the angle BAC′ is equal, not to the proposed angle A, but to $180^\circ - A$.

The triangle BAC′ is consequently distinguished, by the data, from the triangle BAC.

When, therefore, the given side which is opposite to the given angle is greater than the given side which is adjacent to the given angle, the second Case of Art. 59. is not ambiguous.

4th. Art. 59., Case II., Ex. 2., in which $A = 41^\circ\ 36'$, $a = 248$, $c = 354$, are given to find B, C, b.



Construction.—Draw the indefinite line AD: at the point A, make an angle DAB $=41°\,36'$: also make AB $=354$; and from B as centre with a radius $=248$, describe the circumference of a circle.

Then, since BA is greater than BC or a, the point A falls without the circumference of the circle described from the centre B with the radius BC: the points C, C′, in which the circumference cuts AD, are consequently on the same side of the point A. Let BC, BC′, be joined.

By this construction,

in the triangle ABC, A $=41°\,36'$; BC or $a=248$; AB or $c=354$:
And in the triangle ABC′, A $=41°\,36'$; BC′ or $a=248$; AB or $c=354$.

If it is known that, in the proposed triangle, the angle C is acute, ABC is the triangle required; or, if it is known that the angle C is obtuse, ABC′ is the triangle required, (see Euc. VI. 7.): but if it is not known whether C is acute or obtuse, each of the triangles ABC, ABC′, fulfils the conditions of the question.

The values of the angles and side, measured in the usual manner, are

$$\text{ABC}=67°;\ \text{ACB}=71°\tfrac{1}{3};\ \text{AC}=343.8;$$
$$\text{ABC}'=29°\tfrac{1}{2};\ \text{AC}'\text{B}=108°\tfrac{1}{2};\ \text{AC}'=185.5.$$

If a is equal to BC″, the distance of the point B from the line AD, the circumference described from B with the radius a touches the indefinite line AD: the angle BC″A is a right angle, and the side AC″ or b and the angle B have each only one value.

If a is less than the distance from B to the straight line AD, the problem is impossible.

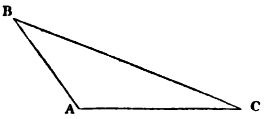
From this discussion of the second Case of Art. 59., it follows:

That if the side opposite to the given angle is greater than the side adjacent to the given angle; or (the side opposite to the given angle being less than the side adjacent to the given angle) if the angle opposite to the greater of the given sides is known to be an acute angle, or an obtuse angle; or, if the less side is equal to the perpendicular let fall from the extremity of the greater of the given sides on the third side of the triangle; the case is not ambiguous.

But, that if the side opposite to the given angle is less than the side adjacent to the given angle, and greater than the perpendicular let fall from the extremity of the greater side on the third side of the triangle, and it is not known whether the angle which is opposite to the greater of the two given sides is acute or obtuse; the case is ambiguous.

5th. Art. 59., Case IV., Ex. 1.; in which $a=800$; $b=562$; $c=320$, are given to find A, B, C.

Construction.—Take from a scale of equal parts the distances 800, 562, 320; and with these three distances, as sides, construct the triangle ABC, (Euc. I. 22.) ABC is the triangle required: and if the angles are measured with a scale of chords or a protractor, it will be found that $A=128°$; $B=33°\frac{1}{2}$; $C=18°\frac{1}{2}$; nearly.



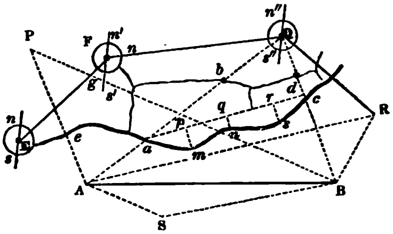
12. OF SURVEYING WITH A THEODOLITE.

THE survey of a tract of ground consists in determining the relative positions of the remarkable points in it, the situations and forms of buildings, the directions of roads and rivers, the boundaries of woods, fields, &c., on the supposition that all are projected on a horizontal plane. The objects imagined to be so projected are represented on paper; and sometimes, both for civil and military purposes, horizontal contour lines of the natural features of the country are determined and introduced on the plan.

In order to find, by means of the theodolite, the positions of the principal objects, as the churches, mills, &c., a base having been measured in the manner described Art. 60., the instrument properly adjusted is set up successively at its two extremities, and the angles contained between this line and others supposed to be drawn from thence to such principal objects as are visible are observed. If possible, the instrument should be placed at each of those objects, and the angles between the lines last mentioned should be observed: the equality of the three observed angles of each triangle to 180 degrees is a proof of the accuracy of the observations.

Since a building, when used as an object or station, does not generally permit the theodolite to be placed upon it accurately at the spot which constitutes an angular point of a triangle, it is customary either to observe the angle with the instrument at some point near the object, afterwards reducing it (Astron. and Geod., Art. 395.) to the value which it would have at the proper station-mark, or to plant vertically in the ground near such object a picket or pole as an angular point, placing the instrument over the spot when the angle is to be observed there; the position of this spot being correctly determined, that of the building with respect to it may be ascertained by measurement.

If AB represent the measured base, and the positions of several objects, as P, Q, R, S, &c., are determined by the instrument in the manner just described, the lines imagined to join them may be considered as new bases, by means of which the positions of other objects may be found in like manner. Frequently also objects are determined by the methods indicated in the Treatise on Astronomy and Geodesy, Arts. 424—426., or by the method of *cross-bearings* indicated in that work, Art. 364.; the principal distances AP, AQ, PQ, &c., may then be computed by the rules of plane trigonometry. In order to find the positions of intermediate points, it is convenient to measure from one of the principal stations, as A, distances, towards another, as Q, to places, as *a*, *b*, &c., where the station-line crosses the streams, roads, or hedges, leaving pickets driven into the ground at all such places, in order that the points may be afterwards distinguished. The like should be done, as at *c*, *d*, *e*, &c., upon the other station-lines; and from these points, by *offsets* or otherwise, new points may be fixed till as many have been found as will serve for tracing the form of the river, road, or boundary line which is to be represented. In executing these measurements, the chain should be kept as much as possible in a horizontal position, being stretched for this purpose in a rectilinear direction over the small inequalities of the ground; and should the ground have a general inclination, the chain may be supported at intervals in a horizontal position, the point on the lower ground, vertically under its extremity, being determined by a plummet suspended from that extremity.



In order to obtain the figure of an irregular line, as *amnsc* between any two fixed points, as *a*, *c*, by means of offsets, the points *p*, *q*, *r*, where perpendiculars from the bends *m*, *n*, *s* meet the line *ac* must be found by estimation or by means of a simple instrument having *sights* at right angles to one another; then the distances *ap*, *pq*, &c., as well as the perpendiculars *pm*, *qn*, &c., being measured, the situations of *m*, *n*, &c., will be determined. By offsets also, the points *m*, *n*, &c., may be transferred to paper on which the line *ac* has been already laid.

When it is required to obtain the contour of a piece of ground which it may be impossible to pass over, as when it is covered with an impenetrable wood, it is necessary to make the survey by going with the instrument quite round the boundaries of the ground; and when it is required to survey a line of road, a river, or a coast, having extensive bends, so that the form cannot be determined with sufficient accuracy by means of offsets from one station-line, the operation must consist in taking short lines nearly coincident with the sinuosities of the object from one extremity to the other: the manner of proceeding in these cases, when a theodolite is used, may be understood from the following description.

Let EF, FQ, QR, represent a series of station-lines along a road, or river, or on part of the contour of a wood, and let E be the first station of the theodolite, the centre of the latter being vertically over that point, *ns* being the direction of the magnetic meridian indicated by the needle in the compass-box of the instrument, and the north point of the needle being at *n*. The optical axis of the telescope is always vertically over a line marked NS in the compass-box; and, at present, let both that axis and the line NS be in a vertical plane passing through the needle *ns*; also let it be supposed that the zero point of the graduations on the horizontal circle of the theodolite, as well as the index of the vernier, is vertically under or over *n*. The lower plate being then fixed to the stand by the clamp-screw, the upper plate with the telescope is turned till the crossing of the wires in the eye-piece of the latter is in apparent coincidence with the object at F; and then the degrees passed over by the index expresses the value of the angle *n*EF, which denotes the bearing of the station-line EF from the magnetic meridian of E.

Now, the upper plate being clamped fast to the lower, let the theodolite be removed, and adjusted with its centre vertically over F; then, turning the whole instrument horizontally, let the crossing of the wires be placed in apparent coincidence with E. In this state, while the needle lies in the direction $n's'$ parallel to *ns*, with its northern point at n', the zero of the graduations will be vertically under s' and the index of the vernier over *g* on the line FE: the angle EFs' should manifestly be equal to the alternate angle *n*EF; and, if the needle coincides with the direction of a line making an angle with NS in the compass-box equal to the observed angle nEF, when the optical axis of the telescope is in the line FE, it is evident that the bearing nEF has been correctly observed.

Next turn the upper plate of the theodolite, the lower one remaining clamped to the stand, till the crossing of the wires in the telescope is in apparent coincidence with the object Q: then the arc passed over by the index will be *gn′h*, but the graduation under the index at *h* will express the arc *s′gn′h;* consequently, 180° being taken from the value of this arc, there will remain the value of the angle n′FQ, or the bearing of FQ from the magnetic meridian n′s′ of F.

The instrument being now placed with its centre vertically over Q, and the upper plate being clamped fast to the lower, turn the whole till the crossing of the wires is in apparent coincidence with F; then, if the needle make an angle with NS in the compass-box equal to the observed angle n′FQ, or its supplement, it will follow that the latter angle has been correctly obtained. Turn next the telescope, from its actual position in the direction QF, till the crossing of the wires is in apparent coincidence with R; then, the zero of the circle being at n″ in the magnetic meridian of Q, the bearing of the line QR from that meridian is expressed by the angle n″QR or its supplement s″QR. The verification of this angle is to be made at R; and in the like manner the operation proceeds to the end of the road or river, or till the circuit of the tract of ground is completed. This process is called surveying by the back-angle; and, supposing A to be the last station, should the bearing of E, the first from the magnetic meridian of A, be equal to the bearing of A from that of E, the work would be said to close well.

When such operations are performed in the field, the observed bearings of the stations with their measured distances from one another, and such offsets as may have been measured on the right or left hand of the station-lines, EF, FQ, &c., to remarkable objects, or bends, are inserted in what is called a *Field-book.* Each page of such book is divided into three columns; and in the middle one, beginning from the bottom, are inserted the successive bearings and distances, while the offsets are written in the columns on the right and left hand, according as they have been measured on the ground.

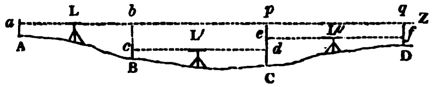
Allotments of land in some of the colonies are usually of a square form; and these are traced on the ground from a base of 6 or 8 miles in length, measured along the bank of a river or on one side of a square previously set out. Such line is divided into parts each equal to one mile, and these into halves and quarters of a mile, the points of division being represented by stakes driven into the ground, or otherwise: at each extremity of the base-line a perpendicular is set out by means of a theodolite, a circumferentor, or a surveying compass, and is divided in like manner. Lines imagined to join the points of division in directions perpendicular and parallel to the base contain within them the several allotments. In order to trace any such line on the ground, which probably is covered with timber, the surveyor determines, by the needle of the instrument, the bearing of the base line from the magnetic meridian passing through the point in it from whence he is to proceed; and adding to that bearing, or subtracting from it 90 degrees, he obtains the bearing of a line perpendicular to it. Directing then the telescope, or the sights of the instrument, so as to make with the direction of the needle the angle thus found, he observes some remarkable tree (small ones, if such there be, in the same line being cut down); and having caused notches to be made in it, in order that it may be recognised, he measures the distance up to it. He then sets up the instrument on the other side of the tree, and having directed the telescope, or the sights, at an angle with the meridian equal to the former, he observes some other tree in that line, measures up to it, and marks it: in the like manner he proceeds to the end of the line, leaving pickets or stakes at every quarter of a mile in its length.

13. OF LEVELLING.

WHEN a spirit-level has been accurately adjusted and is set up vertically over any point on the ground, the optical axis of the telescope being turned round on the vertical axis of the instrument will, in every position, be in a plane which may be considered as a tangent to an imaginary sphere whose centre is that of the earth, and whose surface passes through the centre of the motion; or will be in a plane parallel to one which touches a sphere whose surface passes through the point on the ground: it follows that, if the levelling staff resting on the ground be held vertically at any spots which are at equal distances from the centre of the telescope's revolution, all the points in which the axis of the telescope produced would meet the staff will be at equal distances from the centre or from the surface of the sphere, or of the earth considered as such.

These are called level points; and it is manifest that, if the depressions, or distances from the ground on which the foot of the staff is placed to the points on the latter through which the produced axis of the telescope passes, are not equal to one another, the differences of those depressions, the staves being considered as in parallel directions, will express the depressions of the several points on the ground below a horizontal plane passing through the highest, or their elevations above a horizontal plane passing through the lowest.

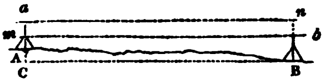
Before a course of levelling is performed it is usual, when great accuracy is required, to have the ground surveyed and pickets driven till their heads are on the level of its surface, at the points at which it is proposed to have the station-staff held up: and it is obvious that the most simple process is to place the spirit-level at or very near the middle of the distance between every two such pickets.



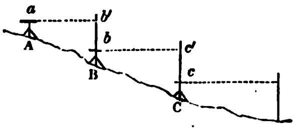
Thus let A, B, C, D, &c., be points at which the staff is to be held up, and L, L′, L″, &c., the telescope of the spirit-level, the axis of the latter being equidistant from each of the two vertical lines passing through the nearest points, but not necessarily in a plane passing through two such lines. Now the line *ab* being the optical axis of the telescope when adjusted between the points A and B, *a* and *b* are points which the upper edge of the sliding vane on the staff passes through when in apparent contact with the crossing of the wires in the telescope; and the difference between the heights A*a*, B*b* expresses (in the figure) the height of A above B. In like manner, the spirit-level being placed midway between the positions B*b* and C*d* of the staff, and *cd* being the direction of the optical axis of the telescope, the difference between B*c* and C*d* expresses the height of B above C. The like differences are taken up to the last station in the line.

It is customary to enter the measured distances *ab*, *cd*, *ef*, &c., in the middle column of a field-book, while in one of the side columns is entered the heights B*b*, C*d*, D*f*, &c., called the fore-readings, and in the other, the heights A*a*, B*c*, C*e*, &c., called the back-readings. By this arrangement, if merely the height of A above D is required, that height will be found by taking the difference between the sum of the fore-readings and the sum of the back-readings. If the profile of the ground is to be represented in the direction of the line from A to D, the following method may be used. Imagine the horizontal line *ab* to be produced indefinitely towards Z, and the vertical lines C*e*, D*f*, &c., to be produced till they meet it in *p*, *q*, &c.: then the heights A*a* and B*b* are given by the operations; C*p* is found by adding the excess of C*d* above B*c* to B*b*; D*q* is found by subtracting the excess of C*e* above D*f* from C*p*; and so on. The distances *ab*, *bp*, &c., being set by any convenient scale upon a line representing *a*Z; perpendicularly to this line make *a*A, *b*B, *p*C, &c., equal to the computed lengths of those lines; then a curved line joining the points A, B, C, D, &c., will represent the required profile. A horizontal plane passing through *a*Z, or any plane parallel to it from which the vertical distances to A, B, C, &c., are reckoned either upwards or downwards, is called a plane of comparison.

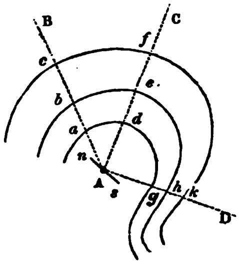
When the inclination of the ground to the horizon is not considerable, it is customary to use a sort of double levelling; thus, let A and B be two stations, *n* the place of the vane on the staff at B when in coincidence with the wires in the telescope at *a*, the level being set up at A, and let *m* be the position of the vane on the staff at A when in coincidence with the wires in the telescope at *b*, the level being set up at B. Then, it being supposed that B*n* and A*m* are perpendicular, and that the visual rays or level lines *an* and *bm* are parallel to the horizon; also, that the heights A*a*, B*b*, of the telescopes from the ground are equal to one another; it is easy to prove, BC being drawn parallel to *an* or *bm*, that B*n*—A*m*, the difference between the two readings, is equal to 2AC, or that half the difference between the readings is equal to the height of A above B. This method is not so convenient as the former, since there is great difficulty in setting up the stand of the level so that the optical axis in the two positions may be at equal heights above the ground.



When the inclination of the surface of the ground to the horizon is great, the stations A, B, C, &c., being very near one another, the instrument must be placed successively at each of those points while the staff is held up at the next below it; and then the differences between the heights A*a*, B*b*, &c., of the instrument and the heights B*b′*, C*c′*, &c., of the sliding vane on the staff are considered as the heights of A above B, B above C, and so on.



This method may be followed when it is required to determine the forms of the contour lines on the features of a country: thus, let AB, AC, AD, be the directions of three vertical planes imagined to cut the rounded extremity of a height; and let it be required to find on the ground the lines in which the surface would be cut by horizontal planes at distances from one another equal, for example, to 3 feet vertically. Setting up the spirit-level at A, and having pickets planted in the ground at B, C, and D, in order to show the directions, fix the vane on the station-staff at a height from its foot equal to 3 feet in addition to the height of the telescope above the ground at A; then let an assistant convey the staff, holding it in a vertical position, from A towards B, till he arrives at a point, as *a*, where the upper edge of the vane is in apparent coincidence with the crossing of the wires in the telescope. The assistant may then proceed to the line AC, and afterwards to AD, moving backwards or forwards in each of those directions till he finds points, as *d* and *g*, at which the vane of the staff will coincide with the crossing of the wires; and the horizontal distances between A and *a*, A and *d*, A and *g*, must then be measured.

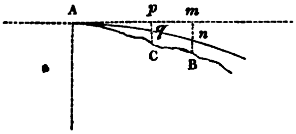


If the station-staff is sufficiently long, the vane may be fixed on it at a distance from its foot equal to 6 feet in addition to the height of the instrument at A; and the assistant placing himself in the directions AB, AC, AD, must move till the vane appears to coincide with the crossing of the wires as before: the horizontal distances *ab*, *de*, *gh*, must then be measured, and pickets must be driven into the ground at *b*, *e*, and *h*.

The spirit-level must now be removed to *b*, and again adjusted; and the vane being fixed on the staff at a distance from its foot equal to 3 feet together with the height of the instrument from the ground at *b*, the assistant must proceed in the direction *b*B, and stop at *c* when the top of the vane is in coincidence with the crossing of the wires; the horizontal distance of *c* from *b* must then be measured. The operations may be continued as before from *b* or *c* as far as necessary towards B: then, commencing at *e* and afterwards at *h*, they may be continued in like manner towards C and D respectively.

The angles which the directions AB, AC, AD, make with the magnetic meridian *ns* passing through A being found by the compass of the theodolite or spirit-level, these directions may be represented on paper: then the measured distances Aa, ab, &c.; Ad, de, &c.; Ag, gh, &c., being set, by the scale of the plan, on those lines of direction, curves drawn through a, d, g; b, e, h; c, f, k, &c., will be the contour lines required.

When the depression Cp of any point C below a horizontal line passing through another point A has been determined by means of a spirit-level at the latter point; if the distance from A to C should be considerable, the surface of an imaginary sphere to which Ap is a tangent, and whose centre is that of the earth, will deviate sensibly from that straight line, and allowance should be made for it. Thus, let mn be the deflection of the spherical surface from Am at any point m; then (Astron. and Geod. Art. 427.), when Am, which may be considered as equal to the arc An, is equal to one mile, mn is equal to about 8 inches; and the deflections mn, pq, &c., at different distances from A, vary nearly as the squares of the distances Am, Ap, &c. Hence, the deflection mn being given, the deflection corresponding to any given distance, as Ap, can be found. Let pq be this deflection; then, if the surface of the ground is, as in the figure, below that of the imaginary sphere, pq must be subtracted from pC, the depression obtained by the spirit-level, in order to have the true depression Cq.



14. OF THE MULTIPLICATION OF DUODECIMAL NUMBERS.

THE area of a parallelogram is obtained by forming the product of numbers which express the values of the base and altitude of the parallelogram; and the volume of a parallelepiped by forming the product of numbers which express the values of the length, breadth, and height of the parallelepiped.

These numbers are frequently expressed in feet, inches, lines, &c.; the principal unit being 1 foot, and 1 of each inferior denomination being $\frac{1}{12}$ part of 1 in that denomination which immediately precedes it. Such numbers are termed duodecimals.

The product of two duodecimal numbers is most readily obtained by reducing them to fractional or decimal expressions of a foot, and performing the multiplication of these expressions by the rule for the multiplication of fractions or decimals.

The product may, however, be obtained without this reduction.

For let $a+\frac{b}{12}+\frac{c}{12^2}+\&c.$; $a'+\frac{b'}{12}+\frac{c'}{12^2}+\&c.$; express two duodecimal numbers: then, if the first of these polynomials is multiplied by the second (Alg. Art. 12.), and all the partial products which have the same denominator are reduced into the same term (Alg. Art. 6.), as under,

$$\begin{array}{rcccccccc}
a + & \frac{b}{12} & + & \frac{c}{12^2} & + & \&c. & & \\
a' + & \frac{b'}{12} & + & \frac{c'}{12^2} & + & \&c. & & \\
\hline
a'a + & \frac{a'b}{12} & + & \frac{a'c}{12^2}. & & & & \\
 & \frac{b'a}{12} & + & \frac{b'b}{12^2} & + & \frac{b'c}{12^3}. & & \\
 & & & \frac{c'a}{12^2} & + & \frac{c'b}{12^3} & + & \frac{c'c}{12^4}. \\
\hline
\end{array}$$

$$a'a+\frac{a'b+b'a}{12}+\frac{a'c+b'b+c'a}{12^2}+\frac{b'c+c'b}{12^3}+\frac{c'c\,.\,.}{12^4}\,.\,.\,.\,,$$

the result is the product of the proposed duodecimal numbers: and since 1 in each term is $\frac{1}{12}$ part of 1 in the preceding term, it is evident that the product is also expressed by a duodecimal number.

Whence, also, the product of two duodecimal numbers being a duodecimal number, if this number is multiplied by a third duodecimal number, the product is again a duodecimal number.

In arranging duodecimal factors for multiplication, the terms of the multiplicand and multiplier which have the same denominator are written in the same vertical column; and this arrangement is also followed with the partial products.

If the left or highest term of the partial product of the multiplicand by each term of the multiplier be written in the vertical column of the term of the multiplier, and the other terms of the partial product, in order, from left to right, it is evident that this arrangement will be obtained; and if the right or lowest term of the partial product of the multiplicand by each term of the multiplier be made to fall one place to the right of the lowest term of the partial product of the multiplicand by the preceding term of the multiplier, that the same arrangement will be obtained. When $a, a', \frac{b}{12}, \frac{b'}{12}$, &c. are replaced by particular numbers, the last is the order which must be followed.

If the factors have the form of compound numbers, the product, also, is written as a compound number.

The first term $(a'a)$ of the product expresses an area, in square feet.

The second term $\left(\frac{a'b + b'a}{12}\right)$ expresses an area in $\frac{1}{12}$ parts of a square foot: and the third term $\left(\frac{a'c + b'b + c'a}{12^2}\right.$ or $\left.\frac{a'c + b'b + c'a}{144}\right)$, in $\frac{1}{144}$ parts of a square foot, or square inches: hence if

the second term is multiplied by 12, and the product added to the third term, the value of the second and third terms is expressed in square inches.

In like manner, since the fourth term expresses an area in $\frac{1}{12}$ parts of a square inch, and the fifth term in $\frac{1}{144}$ parts of a square inch, or square lines, if the fourth term is multiplied by 12 and the product added to the fifth term, the result is the value of the fourth and fifth terms expressed in square lines.

Hence this rule for determining the product of two duodecimal numbers: —

Write the one factor under the other as if for addition of the two numbers:

Form the partial product of the multiplicand by the number which expresses the feet of the multiplier (by the rule for the multiplication of compound numbers), and write the terms of the product under the corresponding terms of the multiplicand: next form the partial product of the multiplicand by the number which expresses the inches of the multiplier; write the first term of this partial product one place to the right of the first term of the preceding partial product, and the other terms in order from right to left:

Again, form the partial product of the multiplicand by the number which expresses the lines of the multiplier; write the first term of the product one place to the right of the product by the number which expresses the inches of the multiplier, and the other terms in order from right to left:

Proceed, in this manner, till all the partial products are formed: then find the sum of the partial products by the rule for the addition of compound numbers: the result is the product of the proposed numbers, also expressed duodecimally.

In the same manner may the product of two duodecimal numbers be multiplied by a third duodecimal number.

The result is a number expressed duodecimally.

If the number which expresses the feet of the multiplier is 0, the partial product of the multiplicand by the number which expresses the inches of the multiplier must still be made to occupy its proper place by writing the first term one place to the right of the lowest denomination of the multiplicand; if the number which expresses the inches of the multiplier is 0, the first term of the partial product of the multiplicand by the number which expresses the lines of the multiplier must be written two places to the right of the lowest denomination of the multiplicand; and so on.

Examples. 1st. Find the product of the duodecimal numbers $3\,.\,\overset{\text{12ths.}}{5}\,.\,\overset{\text{144ths.}}{7}$. and $5\,.\,\overset{\text{12ths.}}{0}\,.\,\overset{\text{144ths.}}{8}$.

$$\begin{array}{rrrrr} & \text{12ths.} & \text{144ths.} & & \\ 3\,. & 5\,. & 7 & & \\ 5\,. & 0\,. & 8 & & \\ \hline 17\,. & 3\,. & 11 & & \\ & 2\,. & 3\,. & 8\,. & 8 \\ \hline 17\,. & 6\,. & 2\,. & 8\,. & 8. \\ \hline \end{array}$$

If the factors of this product express the numbers of feet, inches, and lines in the base and altitude of a parallelogram, the product expresses the area of the parallelogram in square feet, 12th parts of square feet, 144th parts of square feet, &c.

Also if the second term is multiplied by 12 and the third term added to the product, the result, $6 \times 12 + 2$ or 74, expresses a number of square inches: and if the fourth term is multiplied by 12 and the fifth term added to the product, the result, $8 \times 12 + 8$ or 104, expresses a number of square lines. Whence the area of the parallelogram, expressed in square feet, square inches, and square lines, is 17 feet, 74 inches, 104 lines.

2d. Find the product of the duodecimal numbers 4 . $\overset{\text{12ths.}}{3}$. $\overset{\text{144ths.}}{7}$; 2 . $\overset{\text{12ths.}}{6}$. $\overset{\text{144ths.}}{9}$.; and $\overset{\text{12ths.}}{10}$. $\overset{\text{144ths.}}{8}$.

$$\begin{array}{rrrrrrr}
 & \text{12ths.} & \text{144ths.} & & & & \\
4\,. & 3\,. & 7 & & & & \\
2\,. & 6\,. & 9 & & & & \\
\hline
8\,. & 7\,. & 2 & & & & \\
2\,. & 1\,. & 9\,. & 6 & & & \\
 & 3\,. & 2\,. & 8\,. & 3 & & \\
\hline
11\,. & 0\,. & 2\,. & 2\,. & 3 & & \\
 & 10\,. & 8 & & & & \\
\hline
9\,. & 2\,. & 1\,. & 9\,. & 10\,. & 6 & \\
 & 7\,. & 4\,. & 1\,. & 5\,. & 6\,. & 0 \\
\hline
9\,. & 9\,. & 5\,. & 11\,. & 4\,. & 0\,. & 0. \\
\hline
\end{array}$$

If the factors of this product express the numbers of feet, inches and lines in the length, breadth and height of a parallelepiped, the product expresses the volume of the parallelepiped in cubic feet, 12th parts of cubic feet, 144th parts of cubic feet, &c.

The number of cubic inches in $\frac{9}{12}+\frac{5}{144}+\frac{11}{1728}$ of a cubic foot is obtained thus:—

$\frac{9}{12}\times\frac{12}{12}+\frac{5}{144}=\frac{113}{144}$, and $\frac{113}{144}\times\frac{12}{12}+\frac{11}{1728}=\frac{1367}{1728}$ of a cubic foot $=1367$ cubic inches.

The 5th, 6th, and 7th terms may be, in like manner, reduced to cubic lines; the value of the 5th and succeeding terms is, however, generally expressed in fractional parts of a cubic inch. In this example, the 5th term is $\frac{4}{12}$ or $\frac{1}{3}$ of a cubic inch.

Whence the volume of the parallelepiped, expressed in cubic feet and inches, is 9 cubic feet, $1367\frac{1}{3}$ cubic inches.

THE END.

London :
Printed by A. Spottiswoode,
New-Street-Square.

MATHEMATICAL AND SCIENTIFIC WORKS.

I.

ELEMENTS of ARITHMETIC and ALGEBRA. For the use of the Royal Military College, Sandhurst. By W. Scott, Esq., A.M. and F.R.A.S., Second Mathematical Professor in the Institution. Being the First Volume of the Sandhurst Course of Mathematics. 8vo. 16*s.*

II.

ELEMENTS of GEOMETRY; consisting of the First Four and Sixth Books of Euclid, chiefly from the text of Dr. Robert Simson; with the principal Theorems in Proportion, and a Course of Practical Geometry on the Ground; also, Four Tracts relating to Circles, Planes, and Solids, with One on Spherical Geometry. For the use of the Royal Military College, Sandhurst. By John Narrien, Professor of Mathematics in the Institution. Being the Second Volume of the Sandhurst Course of Mathematics. 8vo., with many diagrams, 10*s.* 6*d.*

III.

PRACTICAL ASTRONOMY and GEODESY, including the Projections of the Sphere, and Spherical Trigonometry. For the use of the Royal Military College, Sandhurst. By John Narrien, F.R.S. and R.A.S., Professor of Mathematics in the Institution. Being the Fifth Volume of the Sandhurst Course of Mathematics. 8vo. 14*s.*

IV.

ILLUSTRATIONS of PRACTICAL MECHANICS. By the Rev. H. Moseley, M.A., Professor of Natural Philosophy and Astronomy in King's College, London; being the First Volume of Illustrations of Science by the Professors of King's College. Second Edition, revised. Fcp. 8vo., with numerous Woodcuts, 8*s.*

V.

A TREATISE on the MECHANICAL PRINCIPLES of ENGINEERING and ARCHITECTURE, By the Rev. H. Moseley, M.A. F.R.S., Professor of Natural Philosophy and Astronomy in King's College, London. 8vo., with Illustrations on wood, 24*s.*

VI.

A TREATISE on MECHANICS. By Capt. Kater and Dr. Lardner. New Edition. Fcp. 8vo., with Vignette Title and 19 Plates, comprising 224 distinct figures, 6*s.*

VII.

AN ELEMENTARY TREATISE on ALGEBRA. By James Thomson, LL.D., Professor of Mathematics in the University of Glasgow. 12mo. 5*s.*

VIII.

CROCKER'S ELEMENTS of LAND SURVEYING. New Edition, corrected throughout, and considerably improved and modernised, by T. G. BUNT, Land Surveyor, Bristol. To which are added Tables of Six-figure Logarithms, superintended by Richard Farley, of the Nautical Almanac Establishment. Post 8vo., with Plan of the Manor of North Hill, Somerset, very numerous Diagrams, a Field-book, Plan of Part of the City of Bath, &c., 12*s.*

IX.

TABLES of SIX-FIGURE LOGARITHMS: containing the Logarithms of Numbers from 1 to 10,000, and of Sines and Tangents for every Minute of the Quadrant and every Six Seconds of the First Two Degrees. With a Table of Constants, and Formulæ for the Solution of Plane and Spherical Triangles. Superintended by RICHARD FARLEY, of the Nautical Almanac Establishment. Post 8vo. 4*s.* 6*d.*

X.

A TREATISE on GEOMETRY, and its Application to the Arts. By Dr. LARDNER. Fcp. 8vo., with Vignette Title, and upwards of 200 Figures, 6*s.*

XI.

AN ENCYCLOPÆDIA of ARCHITECTURE, HISTORICAL, THEORETICAL, and PRACTICAL. By JOSEPH GWILT. 8vo. with upwards of 1000 Engravings on wood, from Designs by J. S. Gwilt, 2*l.* 12*s.* 6*d.*

XII.

A DICTIONARY of SCIENCE, LITERATURE, and ART: comprising the History, Description, and Scientific Principles of every Branch of Human Knowledge: with the Derivation and Definition of all the Terms in general use. Edited by W. T. BRANDE, F.R.S. L. & E.; assisted by J. CAUVIN. The various departments are by gentlemen of eminence in each. 8vo., with Wood Engravings, 3*l.*

XIII.

ENCYCLOPÆDIA of GEOGRAPHY; comprising a complete Description of the Earth: exhibiting its Relation to the Heavenly Bodies, its Physical Structure, the Natural History of each Country, and the Industry, Commerce, Political Institutions, and Civil and Social State of all Nations. By HUGH MURRAY, F.R.S.E.: assisted in Astronomy, &c. by Professor Wallace; Geology, &c. by Professor Jameson; Botany, &c. by Sir W. J. Hooker, Zoology, &c. by W. Swainson, Esq. New Edition, with Supplement brought down to December, 1843; with 82 Maps, drawn by Sidney Hall, and upwards of 1000 other Engravings on wood, from Drawings by Swainson, T. Landseer, Sowerby, Strutt, &c., representing the most remarkable objects of Nature and Art in every region of the Globe. 8vo. 3*l.*

LONDON: LONGMAN, BROWN, GREEN, AND LONGMANS.

39, PATERNOSTER ROW: JUNE, 1845.

A Select Catalogue of
BOOKS ON EDUCATION,

PRINTED FOR

LONGMAN, BROWN, GREEN, AND LONGMANS.

Messrs. LONGMAN & Co. *have just published the following*—

A Grammar of the Latin Language.

By C. G. ZUMPT, Ph. D. Professor in the University of Berlin. Translated from the 9th Edition, and adapted for the use of English students, by LEONHARD SCHMITZ, Ph. D.; with the co-operation of Professor ZUMPT. 8vo. 14s. cloth.

"Thus, beyond all question, is the work of Dr. Schmitz henceforward the authorized version of Zumpt's Latin Grammar; a book which well deserves its great celebrity, and the high esteem in which it is held by the best scholars."—EXAMINER.

Prof. Zumpt's School Latin Grammar.

Translated, and adapted for English Schools, by Dr. SCHMITZ, with the co-operation of Professor ZUMPT. 12mo. [In the press.

Also, preparing for publication,

Prof. Zumpt's Latin Exercises.

Exercises to his Grammar of the Latin Language, by Professor ZUMPT. 12mo.

A Copious English-Latin Lexicon.

Compiled from the best Sources. By the Rev. J. E. RIDDLE, M.A. Author of "A Complete Latin-English Dictionary," &c. &c., and the Rev. T. KERCHEVER ARNOLD, M.A. Rector of Lyndon, and late Fellow of Trinity Coll. Cambridge. [In the press.

A Speaker for the use of Schools.

Selected from Classical Greek, Latin, Italian, French, and English Writers—Demosthenes, Thucydides, Homer, Sophocles, Cicero, Livy, Virgil, Lucretius, Shakspeare, Milton, Burke, Bacon, &c. By the Rev. W. SEWELL, B.D., of Exeter College, Oxford. [In the press.

New Class-Book for Ladies' Schools.

The Modern Poetical Reader and Speaker. Selected principally from the best Modern Poets, by especial Permission of the Authors. By Mrs. Palliser. 12mo. [In the press.

Mrs. Marcet's New Spelling Book.

The Mother's First Book; containing Reading made Easy; and the Spelling Book, in Two Parts. By Mrs. MARCET, Author of "Mary's Grammar," "Conversations on Language," &c. &c. 2d Edition, 12mo. with woodcuts, 1s. 6d.

A New Grammar for Boys, by Mrs. Marcet.

Willy's Grammar: interspersed with Stories, and intended for the use of Young Boys. By Mrs. MARCET, Author of "Mary's Grammar," "The Game of Grammar," &c. 2d Edition, 18mo. 2s. 6d.

"A most useful little book, substantially the same as 'Mary's Grammar,' but adapted to Boys, instead of Girls. The principle of this work is that on which we believe all successful teaching must rest."—SPECTATOR.

The Rev. J. Pycroft's Greek Grammar Practice, and Latin Grammar Practice.

[Vide pp. 2 and 4.

⁂ Parents desirous of grounding their sons for the public schools will find these works highly valuable. They have the approbation of some of the first scholars and instructors. The Rev. J. E. Riddle, author of the Latin Dictionary, has testified to their value to junior classes, to those who would teach themselves, and all who would make progress in a sound acquaintance with Greek and Latin Grammar. With these books, any well-educated lady might teach Latin to herself and her son with more ease than French or Italian in the common way. Words and phrases are taught at the same time as rules and inflections.

ELEMENTARY GREEK WORKS, LEXICONS, GRAMMARS, ETC.

Kühner's Elementary Greek Grammar.

An Elementary Grammar of the Greek Language. By Dr. Raphael Kühner, Co-Rector of the Lyceum at Hanover. Translated by J. H. Millard, St. John's College, Cambridge; late Second Classical Master at Mill Hill Grammar School. 8vo. 9s. cloth.

"It affords me much pleasure to be able to assure you, that your translation may be regarded as a pattern of accuracy and care. I perceive with great satisfaction, from your Preface, that you have thoroughly entered into the ideas which have guided me in the preparation of my grammatical works, that you have rightly estimated them, and have exhibited them in a clear light."
EXTRACT OF A LETTER FROM DR. KÜHNER TO THE TRANSLATOR.

Brasse's Greek Gradus.

A Greek Gradus; or, a Greek, Latin, and English Prosodial Lexicon: containing the Interpretation, in Latin and English, of all words which occur in the Greek Poets, from the Earliest Period to the time of Ptolemy Philadelphus: with the Quantity of the Syllables verified by Authorities; and combining the advantages of a Lexicon of the Greek Poets and a Greek Gradus. For the use of Schools and Colleges. By the late Rev. Dr. BRASSE. To which is added, a Synopsis of the Greek Metres, by the Rev. J. R. Major, D.D. Head Master of King's College School, London. 2d Edition, revised and corrected by the Rev. F. E. J. Valpy, M.A. formerly Head Master of Reading School. 8vo. 15s. cloth.

Giles's Greek and English Lexicon.

A Lexicon of the Greek Language, for the use of Colleges and Schools; containing—1. A Greek-English Lexicon, combining the advantages of an Alphabetical and Derivative Arrangement; 2. An English-Greek Lexicon, more copious than any that has ever yet appeared. To which is prefixed, a concise Grammar of the Greek Language. By the Rev. J. A. GILES, LL.D. late Fellow of C. C. College, Oxon. 2d Edit. with corrections, 1 thick vol. 8vo. 21s. cloth.

, The English-Greek Lexicon, separately. 7s. 6d. cloth.

"In two points it excels every other Lexicon of the kind; namely, in the English-Greek part, and in the Classification of Greek Derivatives under their primitives."—MOODY'S ETON GREEK GRAMMAR.

Pycroft's Greek Grammar Practice.

In Three Parts: 1. Lessons in Vocabulary, Nouns, Adjectives, and Verbs in Grammatical order; 2. Greek, made out of each column for translation; 3. English of the same for retranslation. By the Rev. JAMES PYCROFT, B.A. Trinity College, Oxford; Author of "A Course of English Reading," &c. 12mo. 3s. 6d. cloth.

"The plan is excellent, and will tend greatly to facilitate the acquisition of the two languages. By diligent practice in these lessons and vocabularies, the pupil becomes progressively master of all the difficulties that obstruct his early progress, and gradually attains to a well-grounded knowledge, and consequent relish, of the beauties of the Greek and Latin idioms."—JOHN BULL.

Moody's Eton Greek Grammar in English.

The New Eton Greek Grammar; with the Marks of Accent, and the Quantity of the Penult: containing the Eton Greek Grammar in English; the Syntax and Prosody as used at Eton; also, the Analogy between the Greek and Latin Languages; Introductory Exercises and Lessons: with numerous Additions to the text. The whole being accompanied by Practical and Philosophical Notes. By CLEMENT MOODY, A.M. of Magdalene Hall, Oxford; and Editor of the Eton Latin Grammar in English. 2d Edition, carefully revised, &c. 12mo. 4s. cloth.

Valpy's Greek Grammar.

The Elements of Greek Grammar: with Notes. By R. VALPY, D.D. late Master of Reading School. New Edition, 8vo. 6s. 6d. boards; bound, 7s. 6d.

Valpy's Greek Delectus, and Key.

Delectus Sententiarum Græcarum, ad usum Tironum accommodatus: cum Notulis et Lexico. Auctore R. VALPY, D.D. Editio Nova, eademque aucta et emendata, 12mo 4s. cloth.

KEY to the above, being a Literal Translation into English, 12mo. 2s. 6d. sewed.

Valpy's Second Greek Delectus.

Second Greek Delectus; or, New Analecta Minora: intended to be read in Schools between Dr. Valpy's Greek Delectus and the Third Greek Delectus: with English Notes, and a copious Greek and English Lexicon. By the Rev. F. E. J. VALPY, M.A. formerly Head Master of Reading School. 3d Edition, 8vo. 9s. 6d. bound.

The Extracts are taken from the following Writers:—

Hierocles
Æsop
Palæphatus
Plutarch
Polyænus
Ælian
The Septuagint
St. Matthew
Xenophon
Euripides
Sophocles
Æschylus
Aristophanes
Herodotus
Anacreon
Homer
Tyrtæus
Bion; Moschus
Erycius of Cyzicum
Archytas.

Valpy's Third Greek Delectus.

The Third Greek Delectus; or, New Analecta Majora: with English Notes. In Two Parts. By the Rev. F. E. J. VALPY, M.A. formerly Head Master of Reading School. 8vo 15s. 6d. bound.

⁂ The Parts may be had separately.

PART 1. PROSE. 8vo. 8s. 6d. bound. — PART 2. POETRY. 8vo. 9s. 6d. bound.

Valpy's Greek Exercises, and Key.

Greek Exercises; being an Introduction to Greek Composition, leading the student from the Elements of Grammar to the higher parts of Syntax, and referring the Greek of the words to a Lexicon at the end: with Specimens of the Greek Dialects, and the Critical Canons of Dawes and Porson. 4th Edition, with many Additions and Corrections. By the Rev. F. E. J. VALPY, M.A. formerly Head Master of Reading School. 12mo. 6s. 6d. cloth.

KEY, 12mo. 3s. 6d. sewed.

Neilson's Greek Exercises, and Key.

Greek Exercises, in Syntax, Ellipsis, Dialects, Prosody, and Metaphrasis. To which is prefixed, a concise but comprehensive Syntax; with Observations on some Idioms of the Greek Language. By the Rev. W. NEILSON, D.D. New Edition, 8vo. 5s. boards.—KEY, 3s. boards.

Howard's Introductory Greek Exercises, and Key.

Introductory Greek Exercises to those of Huntingford, Dunbar, Neilson, and others; arranged under Models, to assist the learner: with Exercises on the different Tenses of Verbs, extracted from the Table or Picture of Cebes. By NATHANIEL HOWARD. New Edition, with considerable improvements, 12mo. 5s. 6d. cloth.—KEY, 12mo. 2s. 6d. cloth.

Dr. Major's Greek Vocabulary.

Greek Vocabulary; or, Exercises on the Declinable Parts of Speech. By the Rev. J. R. MAJOR, D.D. Head Master of the King's College School, London. 2d Edition, corrected and enlarged, 12mo. 2s. 6d. cloth.

Evans's Greek Copy-Book.

Γραφευς Δοναξ; sive, Calamus Scriptorius: Copies for Writing Greek in Schools. By A. B. EVANS, D.D Head Master of Market-Bosworth Free Grammar School. 2d Edition, 4to. 5s. cloth.

The use of one Copy-Book is sufficient for securing a firm and clear Greek hand.

Dr. Major's Guide to the Greek Tragedians.

A Guide to the Reading of the Greek Tragedians; being a series of articles on the Greek Drama, Greek Metres, and Canons of Criticism. Collected and arranged by the Rev. J. R. MAJOR, D.D. Head Master of King's College School, London. 2d Edition, enlarged, 8vo. 9s. cloth.

The Rev. J. Seager's Translations.

Bos on the Greek Ellipsis.

Professor Schæffer's Edition, with Notes. 8vo. 9s. 6d. boards.

Hermann's Elements of

the Doctrine of Metres. 8vo 8s. 6d. boards.

Hoogeveen on the Greek

Particles. 8vo. 7s. 6d. boards.

Maittaire on the Greek

Dialects. From the Edition of Sturzius. 8vo. 89s. 6d. boards.

Viger on the Greek

Idioms. Abridged and translated into English, from Professor Hermann's last Edition, with Original Notes. 2d Edition, with Additions and Corrections, 8vo. 9s. 6d. boards.

⁂ The above Five Works may be had in 2 vols. 8vo. £2. 2s. cloth.

Dr. Smith's Classical Student's Manual.

The Classical Student's Manual; containing an Index to every Page, Section, and Note, in Matthiæ's Greek Grammar—Hermann's Annotations to Viger on Idioms—Bos on Ellipses—Hoogeveen on the Particles—and Kuster on the Middle Verb; in which Thucydides, Herodotus, Æschylus, Sophocles, Pindar, Homer's Iliad, Xenophon's Anabasis, and the four Plays of Euripides, edited by Professor Parson, are illustrated with Philological and Explanatory Observations by the Rev. W. Collier Smithers, A.M. Ph.D. (of Queen's College, Oxford), Principal of a School at Greenwich. Intended for Students in the Universities, and the Higher Classes in Schools. 3d Edition, 8vo. 10s. 6d. cloth.

LATIN WORKS, DICTIONARIES, GRAMMARS, ETC.

Arnold and Riddle's English-Latin Lexicon.

[Vide page 1.

⁂ Mr. Riddle and Mr. Arnold having found that each has been engaged for a nearly equal space of time in compiling an English-Latin Dictionary from the same sources, have resolved to publish a joint work, the result of a comparison and revision of their respective labours.

Riddle's Latin Dictionary.

A Complete Latin-English and English-Latin Dictionary; compiled from the best sources, chiefly German. By the Rev. J. E. Riddle, M.A. of St. Edmund Hall, Oxford. 4th Edition, corrected and enlarged, 8vo. 31s. 6d. cloth.

Separately { The English-Latin Dictionariy, 10s. 6d. cloth.
The Latin-English Dictionary, £1. 1s. cloth.

Riddle's Young Scholar's Latin Dictionary.

The Young Scholar's Latin-English and English-Latin Dictionary; being an Abridgment of the above. 4th Edition, corrected and enlarged, square 12mo. 12s. bound.

Separately { The Latin-English Dictionary, 5s. 6d. bound.
The English-Latin Dictionary, 7s. bound.

"From the time that a boy at school commences translation of the simplest kind, derivations should be attended to; and we consider Mr. Riddle's an invaluable book, when compared with other Dictionaries, merely on the ground of its large stock of derivations. In the monotony of early instruction these are, perhaps, the very first things that awaken curiosity and interest; a momentary escape and respite, if only apparent, from the matter in hand, is that for which boys are continually craving; and this may be more advantageously indulged by frequent reference to kindred English words, in which they feel themselves at home, than in any other manner."—Church of England Quarterly Review.

Riddle's Diamond Latin-English Dictionary.

A Diamond Latin-English Dictionary. A Guide to the Meaning, Quality, and right Accentuation of Latin Classical Words. By the Rev. J. E. Riddle, M.A. Royal 32mo. 4s. bound.

Zumpt's Latin Grammars and Exercises (Authorised versions).

Translated, and adapted for English Students, by Dr. Schmitz, with the co-opetion of Professor Zumpt. [Vide page 1.

Pycroft's Latin Grammar Practice.

Latin Grammar Practice: 1. Lessons in Vocabulary, Nouns, Adjectives, and Verbs, in Grammatical Order; 2. Latin, made out of each column, for Translation; 3. English of the same, for Re-translation. By the Rev. James Pycroft, B.A. Trinity College, Oxford; Author of "A Course of English Reading," &c. 12mo. 2s. 6d. cloth.

Valpy's Latin Grammar.

The Elements of Latin Grammar: with Notes. By R. Valpy, D.D. late Master of Reading School. New Edition, with numerous Additions and Corrections, 12mo. 2s. 6d. bound.

Taylor's Latin Grammar.

A Latin Grammar, founded on the Eton, and arranged in a Tabular Form, to facilitate Reference and assist the Memory: with Notes, and an Explanation of the Grammatical and Rhetorical Figures in more general use. By the Rev. G. Taylor, D.C.L. Lecturer of Dedham, and late Master of the Grammar School. 3d edition, 3s. cloth.

Dr. Kennedy's Latin Grammar.

Latinæ Grammaticæ Curriculum; or, a Progressive Grammar of the Latin Language, for the use of all Classes in Schools. By Rev. B. H. Kennedy, D.D. Head Master of Shrewsbury School. 2d Edition, 12mo. 4s. 6d. cloth.

Moody's Eton Latin Grammar in English, &c.

The New Eton Latin Grammar, with the Marks of Quantity and the Rules of Accent; containing the Eton Latin Grammar as used at Eton, the Eton Latin Grammar in English: with important Additions, and easy explanatory Notes. By Clement Moody, M A. of Magdalene Hall, Oxf.: Editor of the Eton Greek Grammar in English. 4th Edit. revised throughout and enlarged, 2s. 6d. cloth.

The Eton Latin Accidence: with Additions and Notes. 2d Edition, 12mo. 1s.

Graham's First Steps to Latin Writing.

First Steps to Latin Writing: intended as a Practical Illustration of the Latin Accidence. To which are added, Examples on the principal Rules of Syntax. By G. F. Graham, Author of "English, or the Art of Composition," &c. 2d Edition, considerably enlarged and improved, 12mo. 4s. cloth.

Valpy's Latin Delectus, and Key.

Delectus Sententiarum et Historiarum; ad usum Tironum accommodatus: cum Notulis et Lexico. Auctore R. Valpy, D.D. New Edition, with Explanations and Directions; and a Dictionary, in which the Genders of Nouns, and the principal parts of Verbs, are inserted. 12mo. 2s. 6d. cloth.

Key; being a Literal Translation. By a Private Teacher. New Edition, carefully revised, and adapted to the alterations in the new edition of the text, by W. R. Burdon, 12mo. 3s. 6d. cloth.

Valpy's Second Latin Delectus.

The Second Latin Delectus; designed to be read in Schools after the Latin Delectus, and before the Analecta Latina Majora: with English Notes. By the Rev. F. E. J. Valpy, M.A. Head Master of the Free Grammar-School, Burton-on-Trent. 2d Edition, 8vo. 6s. bound.

Valpy's First Latin Exercises.

First Exercises on the principal Rules of Grammar, to be translated into Latin: with familiar Explanations. By the late Rev. R. Valpy, D.D. New Edition, with many Additions, 18mo. 1s. 6d. cloth.

In this work it has been endeavoured to give the learner some little knowledge of the elements of things, while he is studying the construction of words. A few general principles of science and morality imprinted on the memory at an early age, will never be erased from the mind, and will often lay the foundation of a substantial fabric of useful knowledge.

Valpy's Second Latin Exercises.

Second Latin Exercises; applicable to every Grammar, and intended as an Introduction to Valpy's "Elegantiæ Latinæ." By the Rev. E. Valpy, B.D. late Master of Norwich School. 6th Edit. 12mo. 2s. 6d. cloth.

Intended as a Sequel to Valpy's "First Exercises;" with which the youthful reader is supposed to be fully acquainted before these Exercises are put into his hands. He will thus be led, by a regular gradation, to Valpy's "Elegantiæ Latinæ," to which these Exercises will be an introduction. The Examples are taken from the purest Latin Writers (chiefly the Historians) in Prose and Verse.

Valpy's Latin Vocabulary.

A New Latin Vocabulary; adapted to the best Latin Grammars: with Tables of Numeral Letters, English and Latin Abbreviations, and the Value of Roman and Grecian Coins. By R. Valpy, D.D. 11th Edition, 12mo. 2s. bound.

Valpy's Elegantiæ Latinæ, and Key.

Elegantiæ Latinæ; or, Rules and Exercises illustrative of Elegant Latin Style; intended for the use of the Middle and Higher Classes of Grammar Schools. With the Original Latin of the most difficult Phrases. By Rev. R. Valpy, D.D. late Master of Norwich School. 11th Edition, corrected, 12mo. 4s. 6d. cloth.

Key, being the Original Passages, which have been translated into English, to serve as Examples and Exercises in the above. 12mo. 2s. 6d. sewed.

Butler's Praxis, and Key.

A Praxis on the Latin Prepositions: being an attempt to illustrate their Origin, Signification, and Government, in the way of Exercise. By the late Bishop Butler. 6th Edition, 8vo. 6s. 6d. boards.

Key, 6s. boards.

An Introduction to the Composition of Latin Verse;

containing Rules and Exercises intended to illustrate the Manners, Customs, and Opinions, mentioned by the Roman Poets, and to render familiar the principal Idioms of the Latin Language. By the late Christopher Rapier, A.B. 2d Edition, carefully revised by Thomas Kerchever Arnold, M.A. 12mo. 3s. 6d. cloth.

Key to the Second Edition. 16mo. 2s. 6d. sewed.

Howard's Introductory Latin Exercises.

Introductory Latin Exercises to those of Clarke, Ellis, Turner, and others: designed for the Younger Classes. By Nathaniel Howard. A New Edition, 12mo. 2s. 6d. cloth.

Howard's Latin Exercises extended.

Latin Exercises Extended; or, a Series of Latin Exercises, selected from the best Roman Writers, and adapted to the Rules of Syntax, particularly in the Eton Grammar. To which are added, English Examples to be translated into Latin, immediately under the same rule. Arranged under Models. By Nathaniel Howard New Edition, 12mo. 3s. 6d. cloth.

Key, 2d Edition 12mo. 2s. 6d. cloth.

Bradley's Exercises, &c. on Latin Grammar.

Series of Exercises and Questions; adapted to the best Latin Grammars, and designed as a Guide to Parsing, and an Introduction to the Exercises of Valpy, Turner, Clarke, Ellis, &c. &c. By the Rev. C. Bradley, Vicar of Glasbury. 4th Edition, 12mo. 2s. 6d. bound.

Bradley's Latin Prosody, and Key.

Exercises in Latin Prosody and Versification. By the Rev. C. Bradley, Vicar of Glasbury, Brecon. 8th Edition, with an Appendix on Lyric and Dramatic Measures, 12mo. 3s. 6d. cloth.

Key, 5th Edition, 12mo. 2s. 6d. sewed.

Turner's Latin Exercises.

Exercises to the Accidence and Grammar; or, an Exemplification of the several Moods and Tenses, and of the principal Rules of Construction: consisting chiefly of Moral Sentences, collected out of the best Roman Authors, and translated into English, to be rendered back into Latin; with references to the Latin Syntax, and Notes. By William Turner, M.A. late Master of the Free School at Colchester. New Edition, 12mo. 3s. cloth.

Beza's Latin Testament.

Novum Testamentum Domini Nostri Jesu Christi, interprete Theodora Beza. Editio Stereotypa, 1 vol. 12mo. 3s. 6d. bound.

Valpy's Epitome Sacræ Historiæ.

Sacræ Historiæ Epitome, in usum Scholarum: cum Notis Anglicis. By the Rev. F. E. J. Valpy, M.A. Head Master of the Free Grammar School, Burton-on-Trent. 7th Edition, 18mo. 2s. cloth.

EDITIONS OF GREEK CLASSIC AUTHORS.

Barker's Demosthenes.

Demosthenes—Oratio Philippica I., Olynthiaca I. II. and III., De Pace, Æschines contra Demosthenem, De Corona. With English Notes. By E. H. Barker. 2d Edit. post 8vo. 8s. 6d. boards.

Major's Euripides.

Euripides. From the Text, and with a Translation of the Notes, Preface, and Supplement, of Porson; Critical and Explanatory Remarks, original and selected; Illustrations and Idioms from Matthiæ, Dawes, Viger, &c.; and a Synopsis of Metrical Systems. By Dr. MAJOR, Head Master of King's College School, London. 1 vol. post 8vo. 24s. cloth.

Sold separately as follow, 5s. each:—

ALCESTIS,
HECUBA, 5th Edit.
MEDEA, 4th Edit.
ORESTES, 3d Edit.
PHŒNISSÆ, 2d Edit.

Brasse's Sophocles.

Sophocles, complete. From the Text of Hermann, Erfurdt, &c.; with original Explanatory English Notes, Questions, and Indices. By Dr. BRASSE, Mr. BURGES, and Rev. F. VALPY. 2 vols. post 8vo. 34s. cloth.

Sold separately as follow, 5s. each:—

ŒDIPUS COLONEUS, 2d Edit.
ŒDIPUS REX, 3d Edit.
PHILOCTETES, 3d Edit.
TRACHINIÆ, 3d Edit.
AJAX, 3d Edit.
ANTIGONE, 2d Edit.
ELECTRA, 2d Edit.

Burges's Æschylus.

Æschylus—The Prometheus: English Notes, &c. By G. BURGES, A.M. Trinity College, Cambridge. 2d Edition, post 8vo. 5s. boards.

Belfour's Xenophon's Anabasis.

The Anabasis of Xenophon. Chiefly according to the Text of Hutchinson. With Explanatory Notes, and Illustrations of Idioms from Viger, &c., copious Indexes, and Examination Questions. By F. CUNNINGHAM BELFOUR, M.A. Oxon. F.R.A.S. LL.D. late Professor of Arabic in the Greek University of Corfu. 4th Edit. with Corrections and Improvements, post 8vo. 8s. 6d. bds.

Barker's Xenophon's Cyropædia.

The Cyropædia of Xenophon. Chiefly from the text of Dindorf. With Notes, Critical and Explanatory, from Dindorf, Fisher, Hutchinson, Poppo, Schneider, Sturtz, and other eminent scholars, accompanied by the editor's comments. To which are added, Examination Questions, and copious Indices. By E. H. BARKER, late of Trinity Coll. Camb. Post 8vo. 9s. 6d. bds.

Stocker's Herodotus.

Herodotus; containing the Continuous History alone of the Persian Wars: with English Notes. By the Rev. C. W. STOCKER, D.D. Vice-Principal of St. Alban's Hall, Oxford; and late Principal of Elizabeth College, Guernsey. A New and greatly Improved Edition, 2 vols. post 8vo. 18s. cloth.

Valpy's Homer.

Homer's Iliad, complete: English Notes, and Questions to first Eight Books. Text of Heyne. By the Rev. E. VALPY, B.D. late Master of Norwich School. 6th Edition, 8vo. 10s. 6d. bound.—Text only, 5th Edition, 8vo. 6s. 6d. bound.

* * *

EDITIONS OF LATIN CLASSIC AUTHORS.

Valpy's Tacitus, with English Notes.

C. Cornelii Taciti Opera. From the Text of Brotier; with an Explanatory Notes, translated into English. By A. J. VALPY, M.A. 3 vols. post 8vo. 24s. bds.

Barker's Tacitus—Germany and Agricola.

The Germany of C. C. Tacitus, from Passow's Text; and the Agricola, from Brotier's Text: with Critical and Philological Remarks, partly original and partly collected. By E. H. BARKER, late of Trinity College, Cambridge. 5th Edition, revised, 12mo. 5s. 6d. cloth.

Valpy's Ovid's Epistles and Tibullus.

Electa ex Ovidio et Tibullo: cum Notis Anglicis. By the Rev. F. E. J. VALPY, M.A. Master of Burton-on-Trent School. 3d Edition, 12mo. 4s. 6d. cloth.

*** This selection is the same as that used at Eton, improved by the introduction of English Notes, designed to explain difficulties in the sense, and to account for deviations.

Bradley's Ovid's Metamorphoses.

Ovidii Metamorphoses; in usum Scholarum excerptæ: quibus accedunt Notulæ Anglicæ et Quæstiones. Studio C. BRADLEY, A.M. Editio Octava, 12mo. 4s. 6d. cloth.

Valpy's Juvenal and Persius.

Decimi J. Juvenalis et Persii Flacci Satiræ. Ex edd. Ruperti et Kœnig expurgatæ. Accedunt, in gratiam Juventutis, Notæ quædam Anglicæ scriptæ. Edited by A. J. Valpy, M.A. 3d Edit. 12mo. 5s. 6d. bd.

The Text only, 2d Edition, 3s. bound.

Valpy's Virgil.

P. Virgilii Maronis Bucolica, Georgica, Æneis. Accedunt, in gratiam Juventutis, Notæ quædam Anglice scriptæ. Edited by A. J. Valpy, M.A. 10th Edition, 18mo. 7s. 6d. bound.

The Text only, 19th Edition, 3s. 6d. bound.

Valpy's Horace.

Q. Horatii Flacci Opera. Ad fidem optimorum exemplarium castigata; cum Notulis Anglicis. Edited by A. J. Valpy, M.A. New Edition, 18mo. 6s. bd.

The same, without Notes. New Edition, 3s. 6d.

⁂ The objectionable odes and passages have been expunged.

The Rev. Canon Tate's Edition of Horace.

Horatius Restitutus; or, the Books of Horace arranged in Chronological Order, according to the Scheme of Dr Bentley, from the Text of Gesner, corrected and improved: with a Preliminary Dissertation, very much enlarged, on the Chronology of the Works, on the Localities, and on the Life and Character of that Poet. By James Tate, M.A. 2d edition, to which is now added, an original Treatise on the Metres of Horace, 8vo. cloth, 12s.

Horace on the Middle System.

Classical Student's Translation of Horace; or, the Works of Quintus Horatius Flaccus, translated for Classical Students, on the Principles of the Middle System of Teaching Classics. By Rev H. P. Haughton, B.A. Rector of Markfield, Leicestershire; Author of "The Middle System of Teaching Classics." Royal 18mo. 8s. cloth.

Barker's Cicero de Amicitia, &c.

Cicero's Cato Major, and Lælius; with English Explanatory and Philological Notes; and with an English Essay on the Respect paid to Old Age by the Egyptians. the Persians, the Spartans, the Greeks, and the Romans. By the late E. H. Barker, Esq. of Trinity College, Cambridge. 6th Edition, 12mo. 4s. 6d. cloth.

Valpy's Cicero's Offices.

M. Tullii Ciceronis de Officiis Libri Tres. Accedunt, in usum Juventutis, Notæ quædam Anglicæ scriptæ. Edited by A. J. Valpy, M.A. Editio Quinta, aucta et emendata, 12mo. 6s. 6d. cloth.

Valpy's Cicero's Twelve Orations.

Twelve Select Orations of M. Tullius Cicero. From the Text of Jo. Casp. Orellius; with English Notes. Edited by A. J. Valpy, M.A. 2d Edition, post 8vo. 7s. 6d. boards

Barker's Cæsar's Commentaries.

C. Julius Cæsar's Commentaries on the Gallic War. From the Text of Oudendorp; with a selection of Notes from Dionysius Vossius, from Drs. Davies and Clarke, and from Oudendorp, &c. &c. To which are added, Examination Questions. By E. H. Barker, Esq. late of Trinity College, Cambridge. Post 8vo. with several Woodcuts, 6s. 6d. boards.

Bradley's Phædrus.

Phædri Fabulæ; in usum Scholarum expurgatæ: quibus accedunt Notulæ Anglicæ et Quæstiones. Studio C. Bradley, A.M. Editio Nona, 12mo. 2s. 6d. cl.

Bradley's Cornelius Nepos.

Cornelii Nepotis Vitæ Excellentium Imperatorum: quibus accedunt Notulæ Anglicæ et Quæstiones. Studio C. Bradley, A.M. Editio Octava, 12mo. 3s. 6d. cl.

Bradley's Eutropius.

Eutropii Historiæ Romanæ Libri Septem· quibus accedunt Notulæ Anglicæ et Quæstiones. Studio C. Bradley, A.M. Editio Duodecima, 12mo. 2s. 6d. cloth.

Hickie's Livy.

The First Five Books of Livy: with English Explanatory Notes, and Examination Questions. By D. B. Hickie, LL.D. Head Master of Hawkshead Grammar School. 2d Edition, post 8vo. 8s. 6d. boards.

WORKS BY THE REV. S. T. BLOOMFIELD, D.D. F.S.A.

Bloomfield's Greek Testament.

The Greek Testament: with copious English Notes, Critical, Philological, and Explanatory. 5th Edition, greatly enlarged, and very considerably improved, in 2 closely-printed volumes, 8vo. with Map of Palestine, £2, cloth.

Bloomfield's College and School Greek Testament.

The Greek Testament: with brief English Notes, Philological and Explanatory. Especially formed for the use of Colleges and the Public Schools, but also adapted for general purposes, where a larger work is not requisite. By the Rev. S. T. Bloomfield, D.D. F.S.A. Vicar of Bisbrooke, Rutland; Editor of the larger Greek Testament, with English Notes; and Author of the Greek and English Lexicon to the New Testament, printed uniform with, and intended to serve as a Companion to, the present work. 3d Edition, greatly enlarged and considerably improved, 12mo. 10s. 6d. cloth.

Bloomfield's Greek Lexicon to the New Testament.

Greek and English Lexicon to the New Testament; especially adapted to the use of Colleges and the higher Classes in the Public Schools, but also intended as a convenient Manual for Biblical Students in general. 2d Edition, greatly enlarged, and very considerably improved. Fcp. 8vo. on a wider paper, 10s. 6d. cloth.

Bloomfield's Greek Thucydides.

The History of the Peloponnesian War, by Thucydides. A New Recension of the Text; with a carefully amended Punctuation; and copious Notes, Critical, Philological, and Explanatory; almost entirely original, but partly selected and arranged from the best Expositors, and forming a continuous Commentary: accompanied with full Indices, both of Greek Words and Phrases explained, and matters discussed in the Notes. Dedicated, by permission, to the Right Hon. and Right Rev. Charles James, Lord Bishop of London. Illustrated by Maps and Plans, mostly taken from actual survey. 2 vols. 8vo. 38s. cl.

Bloomfield's Translation of Thucydides.

The History of the Peloponnesian War. By Thucydides. Newly translated into English, and accompanied with very copious Notes, Philological and Explanatory, Historical and Geographical; with Maps and Plates. 3 vols. 8vo. £2. 5s. boards.

HISTORY, CHRONOLOGY, AND MYTHOLOGY.

Lempriere's Classical Dictionary, abridged

For Public and Private Schools of both Sexes. By the late E. H. Barker, Trinity College, Cambridge. A New Edition, revised and corrected throughout. By J. Cauvin. 8vo. 12s. bound.

⁂ This is the only edition containing all the most recent improvements and additions of Professor Anthon, and other eminent scholars; and it is hoped that it will be distinguished from all other editions of Lempriere, which, though larger in size, contain a vast quantity of matter not calculated to assist the scholar, and which has been purposely expunged from this edition; thus diminishing the expense of the work, without injuring its utility as an elementary school-book. All indelicacies, both in matter and language, have been carefully avoided.

Blair's Chronological and Historical Tables.

From the Creation to the Present Time: with Additions and Corrections from the most Authentic Writers; including the Computation of St. Paul, as connecting the Period from the Exode to the Temple. Under the superintendence of Sir Henry Ellis, K.H. Principal Librarian of the British Museum. Imp. 8vo. 31s. 6d. half-bound morocco.

Mangnall's Questions.—The Only Genuine and Complete Edition.

Historical and Miscellaneous Questions, for the Use of Young People; with a Selection of British and General Biography, &c. &c. By R. Mangnall. New Edition, with the Author's last Corrections, and other very considerable Additions and Improvements. 12mo. 4s. 6d. bound.

"The most comprehensive book of instruction existing, and to be preferred to all the others to which it has served as a model."—Quarterly Review.

The only edition with the Author's latest Additions and Improvements, brought down to the present time, bears the imprint of Messrs. Longman and Co.

Corner's Sequel to Mangnall's Questions.

Questions on the History of Europe: a Sequel to Mangnall's Historical Questions; comprising Questions on the History of the Nations of Continental Europe not comprehended in that work. By Julia Corner. New Edition, 12mo. 5s. bound.

Hort's Pantheon.

The New Pantheon; or, an Introduction to the Mythology of the Ancients, in Question and Answer: compiled for the Use of Young Persons. To which are added, an Accentuated Index, Questions for Exercise, and Poetical Illustrations of Grecian Mythology, from Homer and Virgil. By W. J. Hort. New Edition, considerably enlarged by the addition of the Oriental and Northern Mythology. 18mo. 17 Plates, 5s. 6d. bound.

"Superior to all other juvenile mythologies in form and tendency, and decidedly in the pleasure it gives a child."—Quarterly Review.

Hort's Chronology.

An Introduction to the Study of Chronology and Ancient History: in Question and Answer. New Edition, 18mo. 4s. bound.

Müller's Introductory System of Mythology.

Introduction to a Scientific System of Mythology. By C. O. Müller, Author of "The History and Antiquities of the Doric Race," &c. Translated from the German by John Leitch. 8vo. uniform with "Müller's Dorians," 12s. cloth.

Knapp's Universal History.

An Abridgment of Universal History, adapted to the Use of Families and Schools; with appropriate Questions at the end of each Section. By the Rev. H. J. Knapp, M.A. New Edition, with considerable additions, 12mo. 5s. bound.

Bigland's Letters on the Study of History.

On the Study and Use of Ancient and Modern History; containing Observations and Reflections on the Causes and Consequences of those Events which have produced conspicuous Changes in the aspect of the World, and the general state of Human Affairs. By John Bigland. 7th Edition, 1 vol. 12mo. 6s. bds.

Keightley's Outlines of History.

Outlines of History, from the Earliest Period. By Thomas Keightley, Esq. New Edition, corrected and considerably improved, fcp. 8vo. 6s. cloth; or 6s. 6d. bound.

Sir Walter Scott's History of Scotland.

History of Scotland. By Sir Walter Scott, Bart. New Edition, 2 vols. fcp. 8vo. with Vignette Titles, 12s. cloth.

Cooper's History of England.

The History of England, from the Earliest Period to the Present Time. On a plan recommended by the Earl of Chesterfield. By the Rev. W. Cooper. 23d Edition, considerably improved. 18mo. 2s. 6d. cloth.

Baldwin's History of England.

The History of England, for the use of Schools and Young Persons. By Edward Baldwin, Esq., Author of "The History of Rome," &c. A New Edition, carefully revised and corrected, with Portraits. 12mo 3s. 6d. bound.

Valpy's Elements of Mythology.

Elements of Mythology; or, an Easy History o the Pagan Deities: intended to enable the young to understand the Ancient Writers of Greece and Rome. By R. Valpy, D.D. 8th Edition, 12mo. 2s. bound.

Valpy's Poetical Chronology.

Poetical Chronology of Ancient and English History: with Historical and Explanatory Notes. By R. Valpy, D.D. New Edit. 12mo. 2s. 6d. cloth.

Howlett's Tables of Chronology and Regal Genealogies,

combined and separate. By the Rev. J. H. Howlett, M.A. 2d Edition, 4to. 5s. 6d. cloth.

Riddle's Ecclesiastical Chronology.

Ecclesiastical Chronology; or, Annals of the Christian Church, from its Foundation to the Present Time. To which are added, Lists of Councils and of Popes, Patriarchs, and Archbishops of Canterbury. By the Rev. J. E. Riddle, M.A. 8vo. 15s. cloth.

Tate's Continuous History of St. Paul.

The Continuous History of the Labours and Writings of St. Paul, on the basis of the Acts, with intercalary matter of Sacred Narrative, supplied from the Epistles, and elucidated in occasional Dissertations: with the HORÆ PAULINÆ of Dr. PALEY, in a more correct edition (with occasional notes), subjoined. By J. Tate, M.A. Canon Residentiary of St. Paul's. 8vo. with Map, 13s. cloth.

GEOMETRY, ARITHMETIC, LAND-SURVEYING, ETC.

Narrien's Astronomy and Geodesy.

Practical Astronomy and Geodesy: including the Projections of the Sphere and Spherical Trigonometry. For the use of the Royal Military College, Sandhurst. By John Narrien, F.R.S. & R.A.S. Professor of Mathematics, &c. in the Institution. 8vo. 14s.

Scott's Arithmetic and Algebra.

Elements of Arithmetic and Algebra. For the use of the Royal Military College, Sandhurst. By W. Scott, Esq. A.M. and F.R.A.S. Second Mathematical Professor in the Institution. 8vo. 16s. bound.

Narrien's Elements of Euclid.

Elements of Geometry: consisting of the first four, and the sixth, Books of Euclid, chiefly from the Text of Dr. Robert Simson; with the principal Theorems in Proportion, and a Course of Practical Geometry on the Ground. Also, Four Tracts relating to Circles, Planes, and Solids; with one on Spherical Geometry. For the use of the Royal Military College, Sandhurst. By John Narrien, F.R.S. and R.A.S. Professor of Mathematics, &c. in the Institution. 8vo. with many diagrams, 10s. 6d. bound.

Professor Thomson's Elementary Algebra.

An Elementary Treatise on Algebra, Theoretical and Practical. By James Thomson, LL.D. Professor of Mathematics in the University of Glasgow. 12mo. 5s. cloth.

Crocker's Land Surveying.

Crocker's Elements of Land Surveying. New Edition, corrected throughout, and considerably improved and modernized, by T. G. Bunt, Land-Surveyor, Bristol. To which are added, Tables of Six-figure Logarithms, superintended by Richard Farley, of the Nautical Almanac Establishment. Post 8vo. with Plan of the Manor of North Hill, Somerset, belonging to J. W. Antoni, Esq., numerous Diagrams, a Field-book, Plan of part of the City of Bath, &c. 12s. cl.

Illustrations of Practical Mechanics.

By the Rev. H. Moseley, M.A. Professor of Natural Philosophy and Astronomy in King's College, London. Being the First Volume of Illustrations of Science, by the Professors of King's College. 2d Edition, 1 vol. fcp. 8vo. with numerous Woodcuts, 8s. cloth.

Keith on the Globes, and Key.

A New Treatise on the Use of the Globes; or, a Philosophical View of the Earth and Heavens: comprehending an Account of the Figure, Magnitude, and Motion of the Earth: with the Natural Changes of its Surface, caused by Floods, Earthquakes, &c.: together with the Principles of Meteorology and Astronomy: with the Theory of Tides, &c. Preceded by an extensive selection of Astronomical and other Definitions, &c. &c. By Thomas Keith. New Edit. considerably improved, by J. Rowbotham, F.R.A.S. and W. H. Prior. 12mo. with 7 Plates, 6s. 6d. bound.

In this edition are introduced many new questions relating to the positions of the Sun, Moon, and Planets, for the years 1838, 1839, 1840, 1841, and 1842, respectively.

⁂ The only genuine edition, with the Author's latest Additions and Improvements, bears the imprint of Messrs. Longman and Co.

Key, by Prior, revised by J. Rowbotham, 12mo. 2s. 6d. cloth.

Keith's Trigonometry.

An Introduction to the Theory and Practice of Plane and Spherical Trigonometry, and the Stereographic Projection of the Sphere, including the Theory of Navigation; comprehending a variety of Rules, Formulæ, &c. with their Practical Applications to the Measuration of Heights and Distances, to determine the Latitude by two Altitudes of the Sun, the Longitude by the Lunar Observations and to other important Problems on the Sphere; and on Nautical Astronomy' By Thomas Keith. 7th Edition, corrected by S. Maynard, 8vo. 14s. cloth.

Taylor's Arithmetic, and Key.

The Arithmetician's Guide; or, a complete Exercise Book: for Public Schools and Private Teachers. By W. Taylor. New Edition, 12mo. 2s. 6d. bound.

Key to the same. By W. H. White, of the Commercial and Mathematical School, Bedford. 12mo. 4s. bound.

Molineux's Arithmetic, and Key.

An Introduction to Practical Arithmetic; in Two Parts: with various Notes, and occasional Directions for the use of Learners. By T. Molineux, many years Teacher of Accounts and the Mathematics in Macclesfield. In Two Parts. Part 1, New Edition, 12mo. 2s. 6d. bound.—Part 2, 6th Edit. 12mo. 2s. 6d. bd.

Key to Part 1, 6d.—Key to Part 2, 6d.

Hall's Key to Molineux's Arithmetic.

A Key to the First Part of Molineux's Practical Arithmetic; containing Solutions of all the Questions at full length, with Answers. By Joseph Hall, Teacher of Mathamatics. 12mo. 3s. bound.

Simson's Euclid.

The Elements of Euclid: viz. the First Six Books, together with the Eleventh and Twelfth; also the Book of Euclid's Data. By Robert Simson, M.D. Emeritus Professor of Mathematics in the University of Glasgow To which are added, the Elements of Plane and Spherical Trigonometry; and a Treatise on the Construction of Trigonometical Canon: also, a concise Account of Logarithms. By the Rev. A. Robertson, D.D. F.R.S. Savilian Professor of Astronomy in the University of Oxford. 25th Edition, carefully revised and corrected by S. Maynard, 8vo 9s. bound.—Also.

The Elements of Euclid: viz. the First Six Books, together with the Eleventh and Twelfth. Printed, with a few variations and additional references, from the Text of Dr. Simson. New Edition, carefully corrected by S. Maynard, 18mo. 5s. bound.—Also,

The same work, edited, in the Symbolical form, by R. Blakelock, M.A. late Fellow and Assistant-Tutor of Catherine Hall, Cambridge. New Edit. 18mo. 6s. cloth.

Joyce's Arithmetic, and Key.

A System of Practical Arithmetic, applicable to the present state of Trade and Money Transactions: illustrated by numerous Examples under each Rule. By the Rev. J. Joyce. New Edition, corrected and improved by S. Maynard. 12mo. 3s. bound.

Key; containing Solutions and Answers to all the Questions in the work. To which are added, Appendices, shewing the Method of making Mental Calculations, and a New Mode of Setting Sums in the Early Rules of Arithmetic. New Edition, corrected and enlarged by S. Maynard, 18mo. 3s. bound.

Morrison's Book-Keeping, and Forms.

The Elements of Book-keeping, by Single and Double Entry; comprising several Sets of Books, arranged according to Present Practice, and designed for the use of Schools. To which is annexed, an Introduction to Merchants' Accounts, illustrated with Forms and Examples. By James Morrison, Accountant. New Edition, considerably improved, 8vo. 8s. half-bound.

Sets of Blank Books, ruled to correspond with the Four Sets contained in the above work: Set A, Single Entry, 3s.; Set B, Double Entry, 9s.; Set C, Commission Trade, 12s.; Set D, Partnership Concerns, 4s. 6d.

Morrison's Commercial Arithmetic, and Key.

A Concise System of Commercial Arithmetic. By J. Morrison, Accountant. New Edition, revised and improved, 12mo. 4s. 6d. bound.

Key. 3d Edition, corrected and improved by S. Maynard, Editor of "Keith's Mathematical Works." 12mo. 8s. bound.

Nesbit's Mensuration, and Key.

A Treatise on Practical Mensuration: containing the most approved Methods of drawing Geometrical Figures; Mensuration of Superficies; Land Surveying; Mensuration of Solids; the Use of the Carpenter's Rule; Timber Measure, in which is shewn the method of Measuring and Valuing Standing Timber; Artificers' Works, illustrated by the Dimensions and Contents of a House; a Dictionary of the Terms used in Architecture, &c. By A. Nesbit. 12th Edition, corrected and greatly improved, with nearly 700 Practical Examples and nearly 300 Woodcuts, 12mo. 6s. bound.—Key, 7th Edition, 12mo. 5s. bound.

Nesbit's Land Surveying.

A Complete Treatise on Practical Land Surveying. By A. Nesbit. 7th Edition, greatly enlarged, 1 vol. 8vo. illustrated with 160 Woodcuts, 12 Copperplates and an engraved Field-book, (sewed,) 12s. boards.

Farley's Six-Figure Logarithms.

Tables of Six-figure Logarithms; containing the Logarithms of Numbers from 1 to 10,000, and of Sines and Tangents for every Minute of the Quadrant and every Six Seconds of the first Two Degrees: with a Table of Constants, and Formulæ for the Solution of Plane and Spherical Triangles. Superintended by Richard Farley, of the Nautical Almanac Establishment. Post 8vo. (unenumerated,) 4s. 6d. cloth.

Nesbit's Arithmetic, and Key.

A Treatise on Practical Arithmetic. By A. Nesbit. 3d Edition, 12mo. 5s. bd.

A Key to the same. 12mo. 5s. bound.

A Second Part of the above is in the press.

Balmain's Lessons on Chemistry.

Lessons on Chemistry; for the use of Pupils in Schools, Junior Students in the Universities, and Readers who wish to learn the fundamental Principles and leading Facts. With Questions for Examination, a Glossary of Chemical Terms, and an Index. By William H. Balmain. Fcp. 8vo. 6s. cloth.

Mrs. Lee's Natural History for Schools.

Elements of Natural History, for the use of Schools and Young Persons; comprising the Principles of Classification, interspersed with amusing and instructive Original Accounts of the most remarkable Animals. By Mrs. Lee (formerly Mrs T. E. Bowdich), Author of "Taxidermy," &c. 12mo. with 55 Woodcuts, 7s. 6d. bound.

WORKS FOR YOUNG PEOPLE, BY MRS. MARCET.

Mrs. Marcet's New Spelling Book.

The Mother's First Book; containing the Reading made Easy, and Spelling Book. New Edition, 12mo. 1s. 6d. cloth.

Mrs. Marcet's New Grammar for Boys.

Willy's Grammar, interspersed with Stories, and intended for the use of Young Boys. 2d Edition, 12mo. 2s. 6d. cloth. [See page 1.

Lessons on Animals, Vegetables, and Minerals.

18mo. 2s. cloth.

Conversations on the History of England.

For the Use of Children. 2d Edition, with additions, continuing the History to the Reign of George III. 18mo. 5s. cloth.

Mary's Grammar:

Interspersed with Stories, and intended for the use of Children. 6th Edition, revised and enlarged, 18mo. 3s. 6d. half-bound.

"A sound and simple work for the earliest ages."—Quarterly Review.

The Game of Grammar:

With a Book of Conversations (fcp. 8vo.) shewing the Rules of the Game, and affording Examples of the manner of playing at it. In a varnished box, or done up as a post 8vo. volume in cloth, 8s.

Conversations on Language, for Children.

Fcp. 8vo. 4s. 6d. cloth.

Willy's Stories for Young Children:

Containing The House-Building—The Three Pits The Chalk Pit, The Coal Pit, and The Gravel Pit)—and The Land without Laws. 3d Edit. 18mo. 2s. half-bd.

Willy's Holidays:

Or, Conversations on different Kinds of Governments: intended for Young Children. 18mo. 2s. half-bound.

The Seasons:

Stories for very Young Children. New Editions, 4 vols.—Vol. 1, Winter, 3d Edition; Vol. 2, Spring, 3d Edition; Vol. 3, Summer, 3d Edition; Vol. 4, Autumn, 3d Edition. 2s. each, half-bound.

GEOGRAPHY AND ATLASES.

Butler's Ancient and Modern Geography.

A Sketch of Ancient and Modern Geography. By SAMUEL BUTLER, D.D. late Bishop of Lichfield, formerly Head Master of Shrewsbury School. New Edition, revised by his Son, 8vo. 9s. boards; bound in roan, 10s.

Butler's Ancient and Modern Atlases.

An Atlas of Modern Geography; consisting of Twenty-three Coloured Maps, from a new set of plates, corrected, with a complete Index. By the late Dr. BUTLER. 8vo. 12s. half-bound.—By the same Author,

An Atlas of Ancient Geography; consisting of Twenty-two Coloured Maps, with a complete Accentuated Index. 8vo. 12s. half-bound.

A General Atlas of Ancient and Modern Geography; consisting of Forty-five coloured Maps, and Indices. 4to. 24s. half-bound.

⁂ The Latitude and Longitude are given in the Indices.

The Plates of the present new edition have been re-engraved, with corrections from the government surveys and the most recent sources of information.

Edited by the Author's Son.

Abridgment of Butler's Geography.

An Abridgment of Bishop Butler's Modern and Ancient Geography: arranged in the form of Question and Answer, for the use of Beginners. By MARY CUNNINGHAM. 3d Edition, fcp. 8vo. 2s. cloth.

Butler's Geographical Copy-Books.

Outline Geographical Copy-Books, Ancient and Modern: with the Lines of Latitude and Longitude only, for the Pupil to fill up, and designed to accompany the above. 4to. each 4s.; or together, sewed, 7s. 6d.

Goldsmith's Popular Geography.

Geography on a Popular Plan. New Edit. including Extracts from recent Voyages and Travels, with Engravings, Maps, &c. By Rev. J. GOLDSMITH. 12mo. 14s. 6d.

Dowling's Introduction to Goldsmith's Geography.

Introduction to Goldsmith's Grammar of Geography: for the use of Junior Pupils. By J. DOWLING, Master of Woodstock Boarding School. New Edit. 18mo. 9d. sd.

By the same Author,

Five Hundred Questions on the Maps of Europe, Asia, Africa, North and South America, and the British Isles; principally from the Maps in Goldsmith's Grammar of Geography. New Edition, 18mo. 8d.—KEY, 9d.

Goldsmith's Geography Improved.

Grammar of General Geography; being an Introduction and Companion to the larger Work of the same Author. By the Rev. J. GOLDSMITH. New Edition, improved. Revised throughout and corrected by Hugh Murray, Esq. With New Views, Maps, &c. 18mo. 3s. 6d. bound.—KEY, 6d. sewed.

Mangnall's Geography, revised.

A Compendium of Geography; for the use of Schools, Private Families, &c. By R. MANGNALL. A new Edition, revised and corrected throughout. 12mo. 7s. 6d. bound.

Hartley's Geography, and Outlines.

Geography for Youth. By the Rev. J. HARTLEY. New Edit. (the 8th), containing the latest Changes. 12mo. 4s. 6d. bound.—By the same Author,

Outlines of Geography: the First Course for Children. New Edit. 18mo. 9d. sd.

THE FRENCH LANGUAGE.

Hamel's French Grammar and Exercises, by Lambert.

Hamel's French Grammar and Exercises. A New Edition, in one volume. Carefully corrected, greatly improved, enlarged, and arranged, in conformity with the last edition of the Dictionary of the French Academy, and in conformity with the last edition of the French Grammar of the University of France. By N. Lambert. 12mo. 5s. 6d. bound.

KEY, 4s. bound.

Hamel's French Grammar.—(The Original Edition.)

A New Universal French Grammar; being an accurate System of French Accidence and Syntax. By N. Hamel, Graduate in the University of Caen, and Rector of the Town of L'Aigle, in Normandy. New Edit. greatly improved, 12mo. 4s. bd.

Hamel's French Exercises, Key, and Questions.

French Grammatical Exercises. By N. Hamel. New Edition, carefully revised and greatly improved, 12mo. 4s. bound.

Key, 12mo. 3s. bound.—Questions, with Key, 9d. sewed.

Hamel's World in Miniature.

The World in Miniature; containing a faithful Account of the Situation, Extent, Productions, Government, Population, Manners, Curiosities, &c. of the different Countries of the World: for Translation into French. By N. Hamel. New Edition, corrected and brought down to the present time, 12mo. 4s. 6d. bd.

Sandhurst College French Grammar.

A Summary of French Grammar, for the Use of the Gentlemen Cadets of the Royal Military College, Sandhurst. By Henri Marillier, French Master in the Institution. 3d Edition, 12mo. 3s. cloth.

Tardy's French Dictionary.

An Explanatory Pronouncing Dictionary of the French Language, in French and English; wherein the exact Sound of every Syllable is distinctly marked, according to the method adopted by Mr. Walker, in his Pronouncing Dictionary. To which are prefixed, the Principles of the French Pronunciation, &c. By the Abbé Tardy. New Edit. carefully revised, 12mo. 6s. bound.

* * *

ENGLISH GRAMMARS, READING BOOKS, ETC.

Mrs. Felix Summerly's Mother's Primer.

The Mother's Primer. A Little Child's First Steps in many ways. By Mrs. Felix Summerly. Fcp. 8vo. printed in Colours, with a Frontispiece drawn on Zinc by William Mulready, R.A. 1s. sewed.

The Rev. J. Pycroft's Course of English Reading.

A Course of English Reading, adapted to every Taste and Capacity: with Anecdotes of Men of Genius. By the Rev. James Pycroft, B.A. Trinity College, Oxford, Author of "Latin Grammar Practice," and "Greek Grammar Practice." Fcp. 8vo. 6s. 6d. cloth.

Maunder's Universal Class-Book:

A New Series of Reading Lessons (original and selected) for Every Day in the Year: each Lesson recording some important Event in General History, Biography, &c. which happened on the day of the month under which it is placed; or detailing, in familiar language, interesting facts in Science; also, a variety of Descriptive and Narrative Pieces, interspersed with Poetical Gleanings: Questions for Examination being appended to each day's Lesson, and the whole carefully adapted to practical Tuition. By Samuel Maunder, Author of "The Treasury of Knowledge," &c. 2d Edition, revised, 12mo. 5s. bound.

Lindley Murray's Works.

⁎ The only Genuine Editions, with the Author's last Corrections.

1. First Book for Children, 25th edition, 18mo. 6d. sd.
2. English Spelling-Book, 47th edition, 18mo. 18d. bd.
3. Introduction to the English Reader, 34th edit. 12mo. 2s. 6d. bd.
4. The English Reader, 24th edit. 12mo. 3s. 6d. bd.
5. Sequel to ditto, 7th edit. 12mo. 4s. 6d. bound.
6. English Grammar, 51st edit. 12mo. 3s. 6d. bd.
7. English Grammar abridged, 121st edit. 18mo. 1s. bd.
8. English Exercises, 50th edit. 12mo. 2s. bound.
9. Key to Exercises, 12mo. 2s. bd.
10. Exercises and Key, 46th and 25th editions, in 1 vol. 3s. 6d. bound.
11. Introduction au Lecteur François, 6th edition, 12mo. 3s. 6d. bound.
12. Lecteur François, 6th edit. 12mo. 5s. bound.
13. Library Edition of Grammar, Exercises, and Key, 7th edit. 2 vols. 8vo. 21s. bds.

* * *

First Lessons in English Grammar, New edit. revised and enlarged, 18mo. 9d. bd.

Grammatical Questions, adapted to the Grammar of Lindley Murray: with Notes. By C. Bradley, A.M. 8th Edit. improved, 12mo. 2s. 6d. bd.

Enlarged Edit. of Murray's Abridged English Grammar, by Dr. Giles. 18mo. 1s. 6d. cloth.

Mavor's Spelling Book.

The English Spelling-Book; accompanied by a Progressive Series of easy and familiar Lessons: intended as an Introduction to the Reading and Spelling of the English Language. By Dr. MAVOR. 452d Edition, with various revisions and improvements of Dr. Mavor, legally conveyed to them by his assignment, with Frontispiece by Stothard, and 44 beautiful Wood Engravings, designed expressly for the work, by Harvey. The whole printed in an entirely new type, 12mo. 1s. 6d. bound.

⁂ The only Genuine Edition, with the Author's latest Additions and Improvements, bears the imprint of Messrs. Longman and Co.

Carpenter's Spelling-Book.

The Scholar's Spelling Assistant; wherein the Words are arranged according to their principles of Accentuation. By T. CARPENTER. New Edition, corrected throughout, 12mo. 1s. 6d. bound.

NOTICE.—The only Genuine and Complete Edition of CARPENTER'S SPELLING is published by Messrs. Longman and Co. and Messrs. Whittaker and Co. Any person selling any other edition than the above is liable to action at law, and on discovery will be immediately proceeded against, the whole book being copyright.

Blair's Class-Book.

The Class-Book; or, 365 Reading Lessons: for Schools of either sex; every lesson having a clearly-defined object, and teaching some principle of Science or Morality, or some important Truth. By the Rev. D. BLAIR, New Edition, 12mo. 5s. bound.

Blair's Reading Exercises.

Reading Exercises for Schools; being a Sequel to Mavor's Spelling, and an Introduction to the Class-Book. By the Rev. D. BLAIR. New Edition, corrected, 12mo. 2s. bound.

Smart's English Grammar, and Accidence.

The Accidence and Principles of English Grammar. By B. H. SMART. 12mo. 4s. cloth.

The Accidence separately, 1s. sewed in cloth.

Smart's Practice of Elocution.

The Practice of Elocution; or, a Course of Exercises for acquiring the several requisites o a good Delivery. By H. B. SMART. 4th Edition, augmented, particularly by a Chapter on Impassioned Reading Qualified by Taste, with Exercises adapted to a Chronological Outline of English Poetry. 12mo. 5s. cl.

Graham's Art of English Composition.

English; or, The Art of Composition explained in a series of Instructions and Examples. By G. F. GRAHAM. 2d Edition, revised and corrected. Fcp. 8vo. 7s. cloth.

"Among the many elementary works on English composition, we know of none so admirably adapted to the purpose at which it aims as this."—ATLAS.

Graham's Helps to English Grammar.

Helps to English Grammar; or, Easy Exercises for Young Children Illustrated by Engravings on Wood. By G. F. GRAHAM, Author of "English; or, the Art of Composition." 12mo. 3s. cloth.

"At once exact, fully intelligible and explanatory, and, above all, carefully suited to the capacity of children."—ATHENÆUM.

Aikin's British Poets.

Select Works of the British Poets. From Ben Jonson to Beattie. With Biographical and Critical Prefaces, by Dr. AIKIN. A New Edition, with Supplement, by LUCY AIKIN, containing additional Selections from the Works of Crabbe, Scott, Coleridge, Pringle, Charlotte Smith, and Mrs. Barbauld. Medium 8vo. 18s. cloth.

Aikin's Poetry for Children.

Poetry for Children; consisting of Selections of easy and interesting Pieces from the best Poets, interspersed with Original Pieces. By Miss AIKIN. New Edit. considerably improved, 18mo. with Frontispiece, 2s. cloth.

Bullar's Questions on the Scriptures.

Questions on the Holy Scriptures, to be answered in Writing, as Exercises at School, or in the course of Private Instruction. By JOHN BULLAR. New Edit. 18mo. 2s. 6d. cloth.

Wilson and Ogilvy, Skinner Street, Snowhill, London.